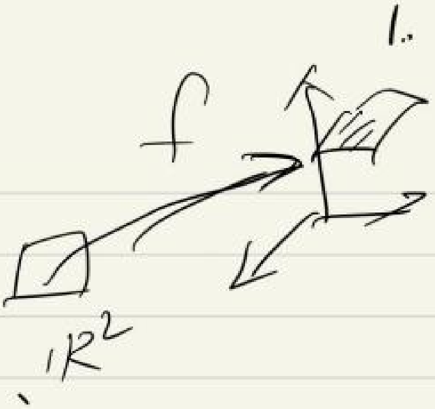


Lecture 1

1. What you should know:

(1) $f: \mathbb{R} \longrightarrow \mathbb{R}$;

(2) $\frac{d}{dx} f$; $\int_a^b f(x)dx$

(3) Fundamental theorem of Calculus.

$$\boxed{\int_a^b f'(x)dx = f(b) - f(a)}$$

What we will learn:

(1) $\vec{f}(\vec{x}): \mathbb{R}^m \longrightarrow \mathbb{R}^n$, $m,n=1,2,3.$;

$$\begin{cases} m=1, \text{ vector-valued function, curve,} \\ m=2, n=3, \text{ surface} \\ m=3, n=3, \text{ field, (electromagnetic field)} \end{cases}$$

(2) $\left(\frac{\partial f_i}{\partial x_j}\right)$ ☆

$\int_D \omega$: ω a differential form;

$\dim D=1$: $f(x)dx$, $f(x,y)dx+g(x,y)dy$, $f(x,y,z)dx+g(x,y,z)dy+h(x,y,z)dz$

$\dim D=2$: $f(x,y)dxdy$

$f(x,y,z)dxdy + g(x,y,z)dydz + h(x,y,z)dxdz.$

$\dim D=3$: $f(x,y,z)dxdydz.$

(3) Green, Gauss, Stokes theorem,

$$\boxed{\int_{\partial D} \omega = \int_D d\omega}$$

$f \longleftrightarrow \omega.$

$[a,b] \longleftrightarrow D.$

$f' \longleftrightarrow d\omega$

$\{a\}\ \{b\} \longleftrightarrow \partial D$

2. What make Calculus III difficult?

- geometry : closed interval ⇝ curves, surfaces, etc.

- linear algebra: constant number ⇝ matrix.

Calculus. 3.

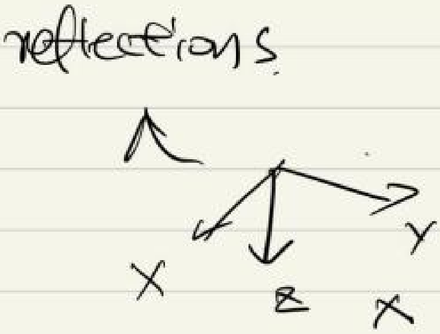
3. $\mathbb{R}^3$.

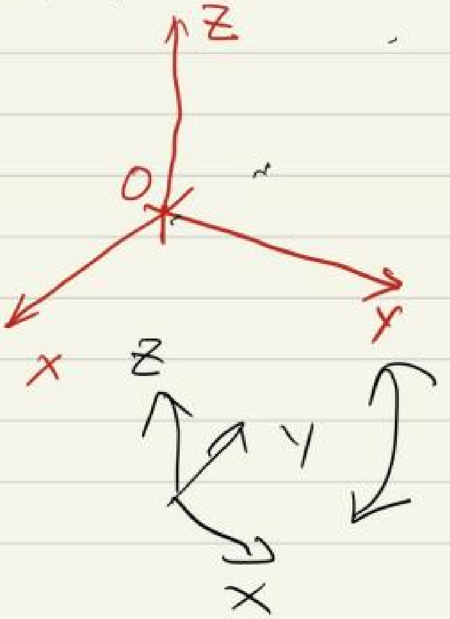
$(a,b) \neq (b,a)$

Any point in the plane can be represented as an ordered pair (a,b) of real numbers.

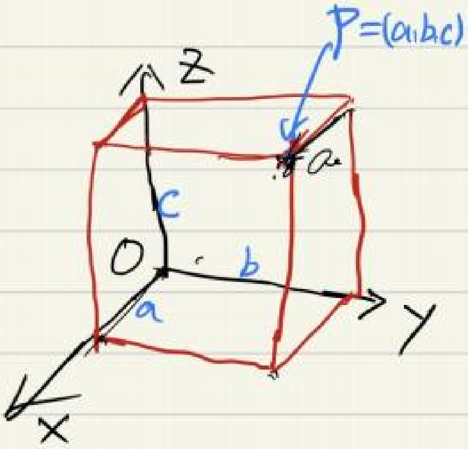
To locate a point in space, three numbers are required. We represent any point in space by an ordered triple (a,b,c) of real numbers.

Geometrically, choose a fixed point O (the origin), and three directed lines through O that are perpendicular to each other, called the coordinate axes.





Now if P is any point in space, let a (resp. b, c), be the directed distance. from the yz (resp. xz, xy) - plane to P. We represent the point P by the ordered triple (a, b, c). of real numbers.

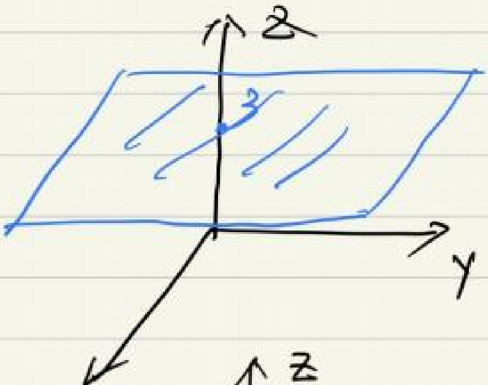


Conversely, to locate the point (a, b, c), draw a rectangular box of directed lengths a, b, c, the P is the farthest point.

4. In three-dimensional analytic geometry, an equation in x, y, z represents a surface in $\mathbb{R}^3$.

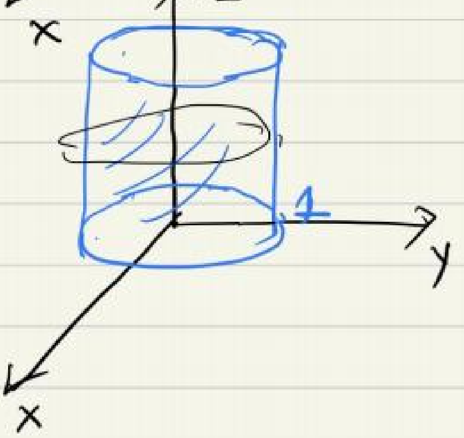
Example 1.

$z=3.$



Example 2

$x^2+y^2=1$

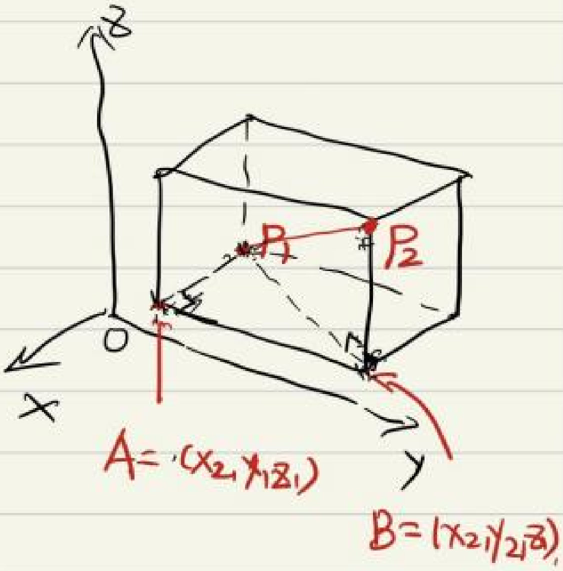


5. Distance and spheres.

- The distance $|P_1P_2|$ between the points $P_1=(x_1,y_1,z_1)$ and $P_2=(x_2,y_2,z_2)$ is.

$$|P_1P_2| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

Proof:



Apply twice the Pythagorean. Theorem. :

$$\left.\begin{aligned} |P_1P_2|^2 &= |P_1B|^2 + |BP_2|^2. \\ |P_1B|^2 &= |P_1A|^2 + |AB|^2 \end{aligned}\right\}$$

$$\Rightarrow |P_1P_2|^2 = \underbrace{\overset{\uparrow}{|P_1A|^2}}_{(x_2-x_1)^2} + \underbrace{\overset{\uparrow}{|AB|^2}}_{(y_2-y_1)^2} + \underbrace{\overset{\uparrow}{|BP_2|^2}}_{(z_2-z_1)^2.}$$

□.

Corollary The An equation of a sphere with center (h,k,l). and radius r is.

$$\boxed{(x-h)^2+(y-k)^2+(z-l)^2=r^2}$$

$f=0 \Longleftrightarrow cf=0$ $c\neq 0$.

Remark: The equation of a sphere, actually any surface is NOT unique; $(f=0)$ and $(cf=0)$ represent the same surface. But to study sphere, we usually "normalize" the equation so that the coefficient of "$x^2+y^2+z^2$" is 1.

Remark: degenerate cases:

- $\underline{(x-h)^2+(y-k)^2+(z-l)^2} = \boxed{0}$ $\longrightarrow$ a point (h,k,l).
- $(x-h)^2+(y-k)^2+(z-l)^2 = c<0$. $\longrightarrow$ ϕ, empty set.

Example 3. Show that

$$x^2+y^2+z^2+4x-6y+2z+6=0$$

is the equation of a sphere, and find its center and radius.

$$x^2+4x+4 + y^2-6y+9 + z^2+2z+1 = -6+4+9+1$$

$$\to \quad (x+2)^2+(y-3)^2+(z+1)^2 = 8 > 0.$$

$(-2,3,-1)$ $r=\sqrt{8}=2\sqrt{2}$

if $6 \rightsquigarrow 15$ ☐

$(x+2)^2+(y-3)^2+(z+1)^2=-1 \rightsquigarrow \phi$

Lecture 2. vectors, dot product, cross product,

1. vectors vs. points.

A vector is a quantity that has both magnitude and direction, (displacement, velocity, force).

A vector is often represented by an arrow or a directed line segment. The length of an arrow represents the magnitude of the vector and the arrow points in the direction of the vector,

Remark when we represent a vector by an arrow, we do not need to specify its source and target.

Therefore, for any vector, we may assume the source is the origin O, then the target is a point P. Conversely, for any point P, $\overrightarrow{OP}$ gives a vector,

In conclusion, there is a one-to-one correspondence between vectors and points in $\mathbb{R}^3$.

We usually do not distinguish a point P and the vector $\overrightarrow{OP}$,

Question: Why do we interpret points as vectors.?

Answer: To define operations.

$\mathbb{R}^3$

2 $\mathbb{R}^3$ as vector space.

Inspired by physics, we have the following definition

Definition If $\bar{u}$ and $\vec{v}$ are vectors positioned so the initial point of $\vec{v}$ is at the terminal point of $\bar{u}$, then the sum $u+v$ is the vector from the initial point of $\bar{u}$ to the terminal point of $\vec{v}$.

In coordinates, if $\vec{u} = (x_1, y_1, z_1)$, $\vec{v} = (x_2, y_2, z_2)$

then $\vec{u}+\vec{v} = (x_1+x_2, y_1+y_2, z_1+z_2)$.

Example $(1,1,1) + \langle 1,2,3\rangle = (2,3,4\rangle$

Definition: If c is a scalar and $\vec{v}$ is a vector, then the scalar multiple $c\vec{v}$ is the vector whose length is $|c|$ times the length of $\vec{v}$ and whose direction is the same as $\vec{v}$ if $c>0$ and is opposite to $\vec{v}$ if $c<0$. If $c=0$ or $\vec{v}=0$, then $c\vec{v}=\vec{0}$.

In coordinates, if $\vec{v} = (x, y, z)$.

then $c\vec{v} = (cx, cy, cz)$,

$c \cdot \vec{v}$

$\vec{w} \cdot \vec{v} \to \mathbb{R}$.

$\vec{w} \times \vec{v} \to \mathbb{R}^3$

Example. $2\langle 1,2,3\rangle = \langle 2,4,6\rangle$,

$-2\langle 1,2,3\rangle = \langle -2,-4,-6\rangle$.

Axioms: 1, $\vec{a}+\vec{b}=\vec{b}+\vec{a}$

2, $\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}$

3, $\vec{a}+\vec{0}=\vec{a}$

4. $\vec{a}+(-\vec{a})=\vec{0}$

5. $c(\vec{a}+\vec{b})= c\vec{a}+c\vec{b}$

6. $(c+d)\vec{a} = c\vec{a}+d\vec{a}$.

7. $(cd)\vec{a} = c(d\vec{a})$.

8. $1\vec{a}=\vec{a}$.

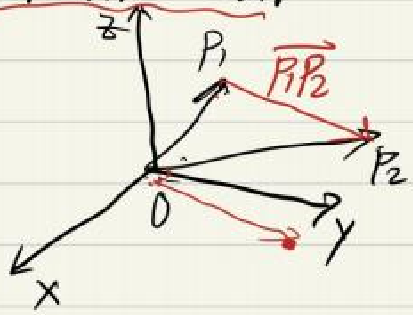
linear algebra.

Remark: These axioms says that $\mathbb{R}^3$, equipped with addition and scalar multiplication, is a vector space

Suppose we have 2 points $P_1=(x_1,y_1,z_1)$, $P_2=(x_2,y_2,z_2)$.

then $\overrightarrow{P_1P_2} = \overrightarrow{OP_2}-\overrightarrow{OP_1} = (x_2-x_1, y_2-y_1, z_2-z_1)$.

$\overrightarrow{OP_1}+\overrightarrow{P_1P_2}=\overrightarrow{OP_2}$



3, Standard basis.

$\vec{i}=(1,0,0)$, $\vec{j}=(0,1,0)$ $\vec{k}=(0,0,1)$.

- Any vector can be uniquely written as a linear combination of $\vec{i}$, $\vec{j}$, and $\vec{k}$.

$$\boxed{(a,b,c) = a\vec{i}+b\vec{j}+c\vec{k}.}$$

$(a,0,0)+(0,b,0)+(0,0,c)=(a,b,c)$

We know the arithmetic of vectors, the next step is the geometry of vectors.

4. lengths of vectors.

①

Definition. If $\bar{v} = (x, y, z)$. then its length.

$$\boxed{\|v\| = \sqrt{x^2+y^2+z^2}}$$

$\|v\|$ can be interpreted as the distance between the initial point and the terminal point.

Example $\vec{v} = (1,1,2)$ $\|v\| = \sqrt{1^2+1^2+2^2} = \sqrt{6}$.

Definition A unit vector is defined as a vector whose length $= 1$.

- To each nonzero vector $\vec{v}$, $\frac{\vec{v}}{\|\vec{v}\|}$ is a unit vector.
- $\vec{0}$ does not produce any unit vector. it has no direction.

Taking the unit vector means that we neglect its length, we only care about its direction.

Example. $\vec{v} = (1,1,2)$ $\frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{6}}(1,1,2)$.

5. Dot product

②

Angles between two vectors is also an important concept describing the relative position of two vectors, This is closely related to the so called dot product

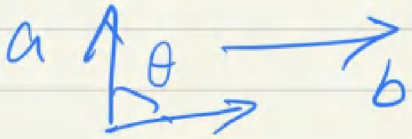
$\mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$

Definition. If $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$, then the dot product of $\vec{a}$ and $\vec{b}$ is the number $\vec{a}\cdot\vec{b}$ given by.

$$\boxed{\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3}$$

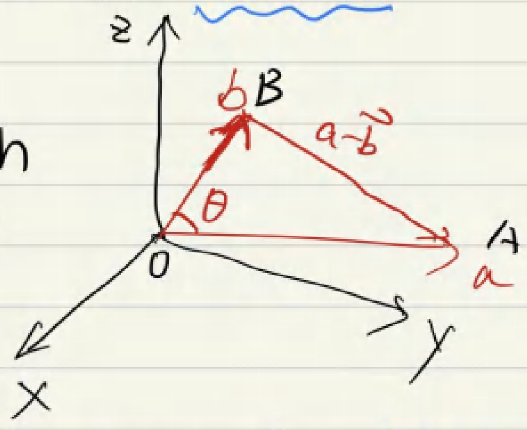
Properties: If $\vec{a}, \vec{b}, \vec{c}$ are vectors and c is a scalar, then.

1. $\vec{a}\cdot\vec{a} = |a|^2 \longleftarrow$ dot product recovers. length.
2. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$
3. $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$
4. $(c\vec{a})\cdot\vec{b} = c(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(c\vec{b})$
5. $\vec{0}\cdot\vec{a} = 0$.



Caution: $(\vec{a}\cdot\vec{b})\cdot\vec{c}$ is NOT defined.

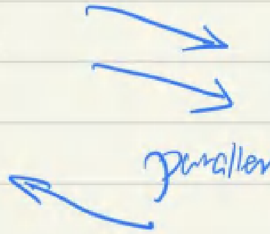
The angle θ between $\vec{a}$ and $\vec{b}$ is defined to be the angle between the representations of $\vec{a}$ and $\vec{b}$ that start at the origin. where $0 \leq \theta \leq \pi$.



Two vectors are called parallel to each other. if they have the same or the opposite directions ($\theta = 0$ or π).

In coordinates, $\vec{0} \neq \vec{v}_1 = (x_1, y_1, z_1)$. is parallel to $0 \neq v_2 = (x_2, y_2, z_2)$ if and only if.

$$\boxed{v_1 = cv_2 \quad \text{for some } c \neq 0.}$$

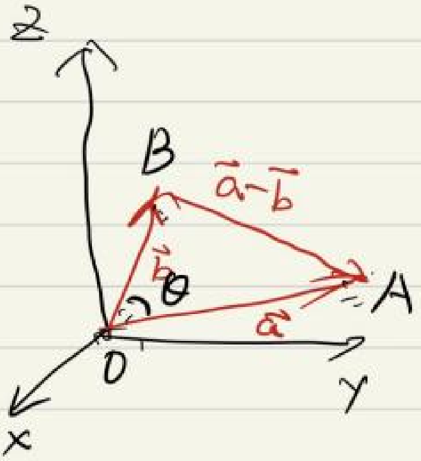


$\vec{0}$ is parallel to any vector, $\nearrow 2(1,2,3)$

Example (1,2,3) is parallel to (2,4,6), (-1,-2,-3), $\searrow -(1,2,3)$
but not parallel to (1,1,1),

Theorem: $\boxed{\vec{a}\cdot\vec{b} = |a|\,|b|\cos\theta}$

Proof: Apply the Law of cosine,



$$|\vec{a}-\vec{b}|^2 = |a|^2+|b|^2-\underbrace{2|\vec{a}|\,|\vec{b}|\cos\theta}_{} \quad (1)$$

But

$$|\vec{a}-\vec{b}|^2 = (\vec{a}-\vec{b})(\vec{a}-\vec{b})$$

$$= \vec{a}\cdot\vec{a}-\vec{a}\cdot\vec{b}-\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}$$

$$= |a|^2+|b|^2-\underbrace{2\vec{a}\cdot\vec{b}}_{}. \quad (2)$$

Comparing (1) and (2) □

Remark: The proof shows that; If we know the lengths of <u>all</u> vectors, we know angles between any pair of vectors.

↑ angles can be deter by lengths

Corollary geometry → $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$. ← algebra

Corollary Two vectors $\vec{a}$ and $\vec{b}$ are orthogonal if and only if $\underline{\vec{a}\cdot\vec{b}=0}$.

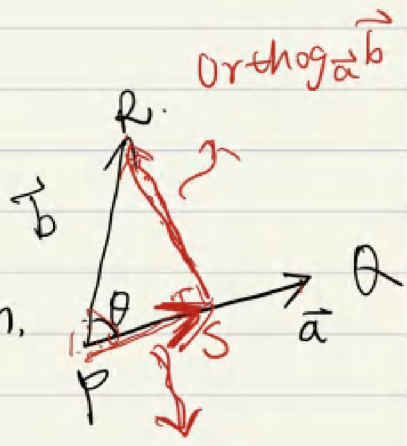
$\uparrow \cos\theta = 0 \Leftrightarrow \theta = \frac{\pi}{2}$

(We assume that $\vec{0}$ is orthogonal to any vector),

Example. (1, 2, 1). (−2, 1, 0). ⟶ orthogonal

$$\vec{a}\cdot\vec{b} = 1\times(-2)+2\times1+0=0$$

6. Projections



The vector projection

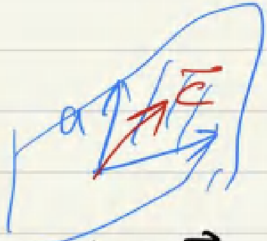
$\mathrm{Proj}_{\vec{a}}\vec{b}$ is defined by the graph.

The scalar projection of $\vec{b}$ onto $\vec{a}$
(also called the component of $\vec{b}$ along $\vec{a}$) is $\mathrm{Proj}_{\vec{a}}\vec{b}$
defined to be the magnitude of the
vector projection. which is the number
$|\vec{b}|\cos\theta$. ⇐ definition of cosθ.

$\mathbb{R}$. ⇒ $\mathrm{Comp}_{\vec{a}}\vec{b} = |b|\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{a}{|\vec{a}|}\cdot\vec{b}.$ $\quad \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{\cancel{|\vec{a}|}\cdot|\vec{b}|}{\cos\theta}$

$\mathbb{R}^3$ ⇒ $\underbrace{\mathrm{Proj}_{\vec{a}}\vec{b}}_{\text{vector.}} = \frac{\vec{a}}{|\vec{a}|}\,\underbrace{\mathrm{comp}_{\vec{a}}\vec{b}}_{\text{length}} = \frac{\vec{a}\cdot\vec{b}}{|a|^2}\,\vec{a}.$

7. Cross Product. unit vector "direction"



☆

Given nonzero vectors $\vec{a}$, $\vec{b}$, if $\vec{a}$ is not parallel to $\vec{b}$, $\vec{a}$. $\vec{b}$ spans a plane, Cross product tells you how to find a vector orthogonal to this plane. to find normal vector.

Definition If $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$ then the cross product of $\vec{a}$ and $\vec{b}$ is the vector.

$$\boxed{\vec{a}\times\vec{b} = \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1\rangle.}$$

☆

Remark Dot product can be defined for any dimensional vector spaces, but cross product is only defined for three-dimensional vectors,

$\mathbb{R}^2$ ⇐ $(x_1, y_1)\cdot(x_2, y_2) = x_1x_2 + y_1y_2$

matrix $\longrightarrow \mathbb{R}$.

matrix

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{2\times 2}}$$

~~$\cong \mathbb{R}^4$~~

Determinant of order 2:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Determinant of order 3: matrix $\longrightarrow \mathbb{R}$.

row →

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \boxed{-} a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

6 terms. $= 2\times 3$.

Formally,

$$\boxed{\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}}$$

symbols

$\vec{i} = (1,0,0)$

$j = (0,1,0)$

$k = (0,0,1)$

linear combinations of $\vec{i}, \vec{j}, \vec{k}$.

, a vector.

Example 1. $\vec{a} = (1,3,4)$. $\vec{b} = (2,7,-5)$.

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i}, & \vec{j}, & \vec{k}, \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix}$$

$$= \underline{-43\vec{i} + \boxed{13}\vec{j} + \vec{k}}.$$

$-(-5-8)$

$7-6=1$

$-15-28=-43$

8 Properties of the cross product.

The definition of $\vec{a}\times\vec{b}$ looks weird, so we need some geometric interpretation,

(1). $\vec{a}\times\vec{a} = 0$. $\longrightarrow$ if $\vec{a}$ $\vec{b}$ paraller

$\downarrow$

$\vec{a}\times\vec{b} = 0$.

By definition, if $\vec{a} = (a_1\ a_2\ a_3)$.

$$\vec{a}\times\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= \underset{0}{(a_2a_3 - a_3a_2)}\vec{i} - \underset{=0}{(a_1a_3 - a_3a_1)}\vec{j} + \underset{=0}{(a_1a_2 - a_2a_1)}\vec{k}.$$

(2) The vector $\vec{a}\times\vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

$$(\vec{a}\times\vec{b})\cdot\vec{a} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3$$

$$= a_1(a_2b_3 - a_3b_2) - a_2(a_1b_3 - a_3b_1) + a_3(a_1b_2 - a_2b_1)$$

$$= 0.$$

□

(3) If θ is the angle between $\vec{a}$ and $\vec{b}$ $(0 \le \theta \le \pi)$, then

$$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta.$$

$$|\vec{a}\times\vec{b}|^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$$

$$= a_2^2b_3^2 - 2a_2a_3b_2b_3 + a_3^2b_2^2$$

$$+ a_3^2b_1^2 - 2a_1a_3b_1b_3 + a_1^2b_3^2$$

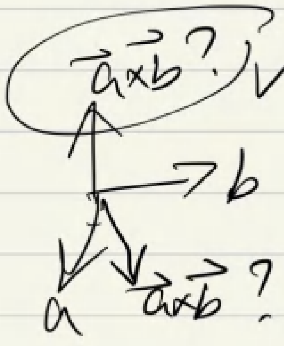
$$+ a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2$$

$= (a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2) - (a_1b_1+a_2b_2+a_3b_3)^2$

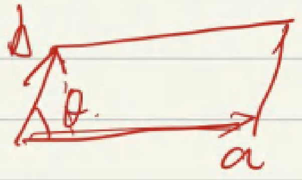
$= |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2$

$= |\vec{a}|^2|\vec{b}|^2 - |\vec{a}|^2|\vec{b}|^2\cos^2\theta.$

$= |\vec{a}|^2|\vec{b}|^2\sin^2\theta$ □.



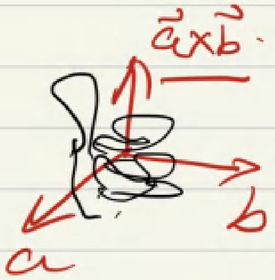
Remark: $|\vec{a}\times\vec{b}|$ is the area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$



So: $\vec{a}\times\vec{b}$ is ① orthogonal to $\vec{a}$ and $\vec{b}$, of length $|\vec{a}||\vec{b}|\sin\theta$, what about the direction?
②

③ right-hand-rule:

it the fingers of your right hand curl in the direction of a rotation. from $\vec{a}$ to $\vec{b}$, the your thumb points in the direction of $\vec{a}\times\vec{b}$,



Corollary. Two nonzero vectors $\vec{a}\times\vec{b}$ are parallel if and only if

$$\boxed{\vec{a}\times\vec{b}=\vec{0}.}$$

Caution Cross product is NOT commutative,

$\vec{i}\times\vec{j}=\vec{k}$, $\vec{j}\times\vec{i}=-\vec{k}$, cross product is NOT associative.

$\underline{\vec{i}\times(\vec{i}\times\vec{j})} = \vec{i}\times\vec{k} = -\vec{j}$.　　$\times : \mathbb{R}^3\times\mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$\underline{(\vec{i}\times\vec{i})\times\vec{j}} = \vec{0}\times\vec{j} = \vec{0}$.　　$\boxed{\vec{a}\times\vec{b}\times\vec{c}}\ \times \leftarrow$ numbe

Remark: $(\mathbb{R}^3, \times)$ is NOT an associative, but it is. a Lie algebra.

(5) Further properties:

$\vec{a}, \vec{b}, \vec{c}$ are vectors, c is scalar.

1. $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$.
2. $(c\vec{a})\times\vec{b} = c(\vec{a}\times\vec{b}) = \vec{a}\times(c\vec{b})$.
3. $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$,
4. $(\vec{a}+\vec{b})\times\vec{c} = \vec{a}\times\vec{c} + \vec{b}\times\vec{c}$.
5. $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$.
6. $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$.

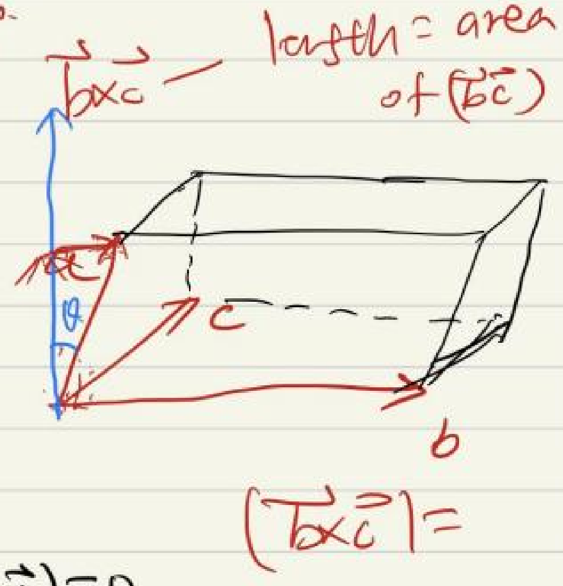
9, Triple Products:

The scalar triple product of the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is defined as $\vec{a}\cdot(\vec{b}\times\vec{c}) \longrightarrow \mathbb{R}$.

$\quad\quad \hookrightarrow$ vector

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{matrix} \leftarrow \vec{a} \\ \leftarrow \vec{b} \\ \leftarrow \vec{c} \end{matrix}$$

Geometric meaning.

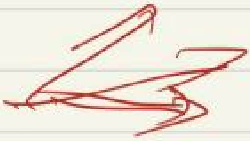


volume of the parallelepiped.

$$V = Ah = |\vec{a}\cdot(\vec{b}\times\vec{c})|$$

Corollary $\vec{a}, \vec{b}, \vec{c}$ is coplanar $\Leftrightarrow$ $\vec{a}\cdot(\vec{b}\times\vec{c}) = 0$

Remark, $\vec{a}\cdot(\vec{b}\times\vec{c}) > 0$ if and only if $\vec{a}, \vec{b}, \vec{c}$ satisfies the right-hand rule.



Example: $\vec{a} = (2,-1,1)$, $\vec{b} = (2,1,1)$, $\vec{c} = (1,2,3)$.

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} 2 & -1 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

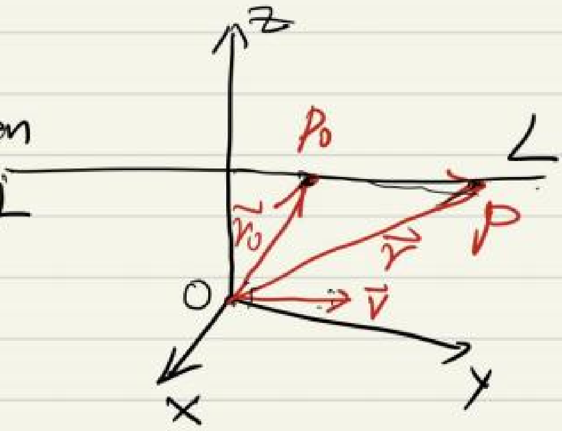
$$= 2 + 5 + 3 = 10.$$

□

Lecture 3. Lines. planes, distances.

1. Lines.

A line in $\mathbb{R}^3$ is determined when we know a point $P_0(x_0, y_0, z_0)$ on L and the direction of L.



Let $\vec{v}$ be a vector parallel to L.

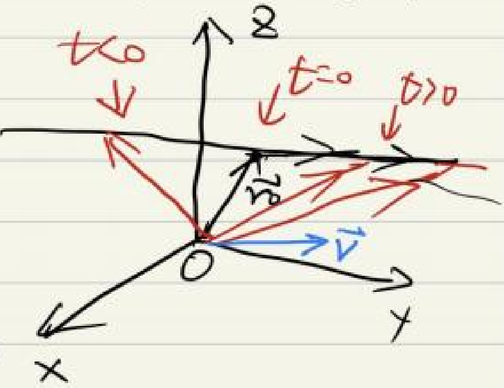
Let P_0 be a point on L. P an arbitrary point on L. $\vec{r_0}$, and $\vec{r}$ the corresponding position vectors.

$\vec{v}$ is parallel to L $\Longleftrightarrow$ $\underline{\overrightarrow{P_0P} = t\vec{v}}$.

so we get.

$$\boxed{\vec{r} = \vec{r_0} + t\vec{v}}$$

As t varies, the line is traced out by the tip of the vector $\vec{r}$.



if $t=0$ $\vec{r} = \vec{r_0}$

$t>0$. half line one direction

$t<0$ opposite direction

If we write everything in coordinates,

$$\vec{v} \longmapsto (a,b,c).$$
$$t\vec{v} \longrightarrow (ta, tb, tc)$$
$$\vec{r_0} \longrightarrow (x_0, y_0, z_0). \leftarrow \text{fixed point}$$
$$\vec{r} \longrightarrow (x, y, z). \leftarrow \text{arbitrary point}$$

we get

①

Definition Parametric equations for a line through the point $P_0(x_0, y_0, z_0)$, and parallel to the direction vector (a,b,c) are.

$$\boxed{x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct}$$

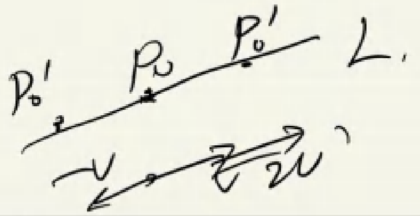
Remark: These equations determine a vector-valued function

$$\mathbb{R} \longrightarrow \mathbb{R}^3$$
$$t \longmapsto (x(t), y(t), z(t)).$$

$t \longrightarrow$ point / vector

Example If $P_0 = (5,1,3)$, $\vec{v} = (1,4,-2)$, then parametric equations are

$$x = 5 + t \quad y = 1 + 4t, \quad z = 3 - 2t$$



Remark The vector equation and parametric equations of a line are not unique,

- the direction vector $\vec{v}$ is only determined up to a nonzero constant: $\vec{v}$ and $c\vec{v}$ represent the same direction.

- Once the direction is chosen, any point P on the line L can be chosen to get the equation for the same line L.

Roughly speaking, the set of all lines in the space is a 4-dimensional object (manifold). ☆

$\longrightarrow 3+3-2$!?

Another way of describing a line L is to eliminate the parameter t from the parametric equations.

(1) If none of the a, b, c is 0,

$$t = \frac{x-x_0}{a}, \quad t = \frac{y-y_0}{b}, \quad t = \frac{z-z_0}{c}.$$

$\Rightarrow$ ②

$$\boxed{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}}$$

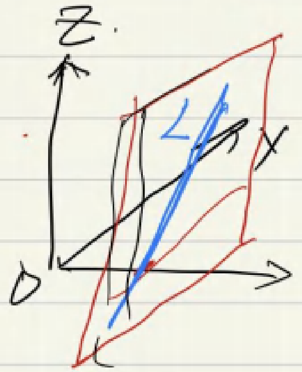
These equations are called symmetric equations of L.

$$\begin{cases} \frac{x-x_0}{a} = \frac{y-y_0}{b} & \Leftarrow \text{linear equation of } x, y \quad \text{plane in } \mathbb{R}^3. \\ \frac{y-y_0}{b} = \frac{z-z_0}{c} & \longrightarrow \text{plane in } \mathbb{R}^3 \end{cases}$$

(2) If one of a, b, c is 0, say, $a = 0$.

we could write the equation of L as.

$$x = x_0, \qquad \frac{y-y_0}{b} = \frac{z-z_0}{c}.$$

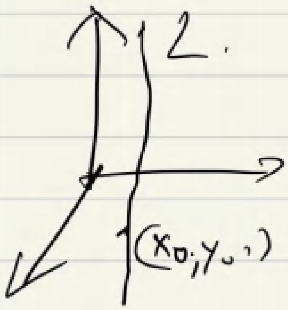


L is in a plane parallel to some coordinate planes.

(3) If two of a, b, c are 0, say, $a = b = 0$,

then,

$$x = x_0, \qquad y = y_0,$$



L is parallel to a coordinate axes.

Remark: Essentially, the symmetric equations of L represents L as the intersection of two planes.

Remark Of course, the symmetric equations of L are NOT unique

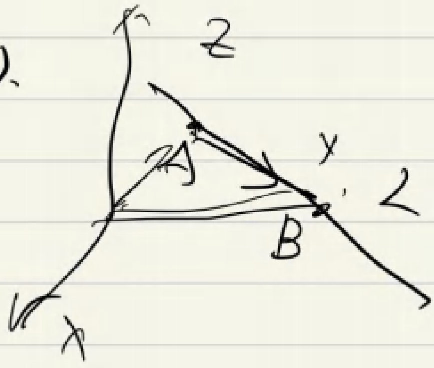
Example Find the symmetric equations of the line that passes through the points A(2,4,−3) and B(3,−1,1).

- direction. $\vec{v} = \overrightarrow{AB} = (1,-5,4)$.
- a point $A(2,4,-3)$.

$$\Rightarrow \quad \frac{x-2}{1} = \frac{y-4}{-5} = \frac{z+3}{4}.$$

↑

NOT unique,



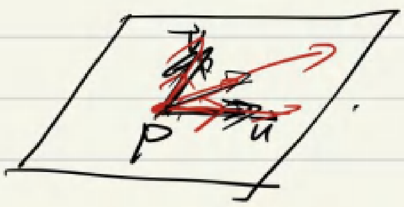
1 curve, line 1.parameter. t. 5,

2. surface, plane 2 parameter. s.t.

2. Planes

It is natural to describe a plane as follows:

Find a point P in the plane, and two non-parallel vectors $\vec{u}$ and $\vec{v}$, in the plane,



the.

$$P + r\vec{u} + s\vec{v}. \quad (r,s\in\mathbb{R})$$

Let, r,s vary

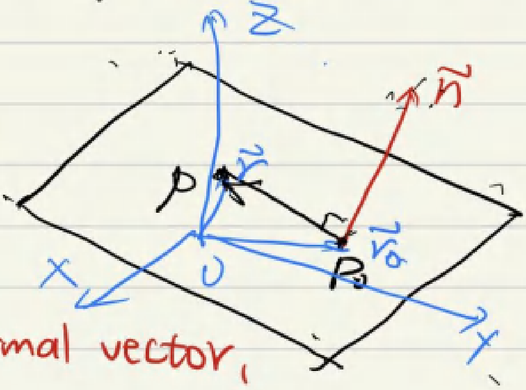
represents the plane,

Actually, we can always represent a surface as a map

$$\mathbb{R}^2 \longrightarrow \mathbb{R}^3,$$

$$\underline{(s,t)} \longmapsto (x(s,t), y(s,t), z(s,t)), \qquad \square$$

Fortunately, there is a simpler way to represent a plane in $\mathbb{R}^3$.



A plane in space is determined by a point $P_0(x_0,y_0,z_0)$ in the plane and a vector $\vec{n}$ that is orthogonal to the plane, normal vector,

Let $P(x,y,z)$ be an arbitrary point in the plane. let $\vec{r_0}$ and $\vec{r}$ be the position vectors of P_0 and P, the $\overrightarrow{P_0P}$ is orthogonal to $\vec{n}$

$$\Rightarrow \quad \boxed{\vec{n}\cdot(\vec{r}-\vec{r_0})=0} \quad \longleftarrow \text{orthogonal}$$

Equivalently,

$$\boxed{\vec{n}\cdot\vec{r} = \vec{n}\cdot\vec{r_0}}$$

linear equation of (x,y,z)

$ax+by+cz$

constant,

d.

Write everything in coordinates,

$$\vec{n} \rightsquigarrow (a,b,c).$$

$$\vec{r} \longrightarrow (x,y,z) \leftarrow \text{arbitrary point} \leftarrow \text{free variable}$$

$$\vec{r_0} \longrightarrow (x_0,y_0,z_0). \leftarrow \text{constant}$$

Definition A scalar equation of the plane through point $P_0(x_0,y_0,z_0)$ with normal vector, $\vec{n} = \langle a,b,c\rangle$, is

$$\boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0}$$

$\underbrace{(x, y, z)}$ related by an equation

Example. $P_0 = (2,4,-1)$. $\vec{n} = (2,3,4)$.

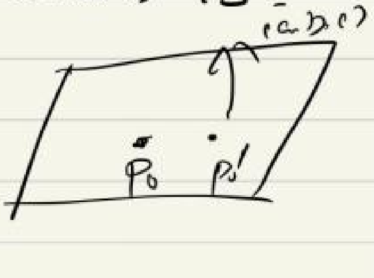
$$\Rightarrow \quad 2(x-2) + 3(y-4) + 4(z+1) = 0.$$

$$\Longleftrightarrow \quad 2x + 3y + 4z = 12.$$

Definition A linear equation of the plane through point $P_0(x_0,y_0,z_0)$ with normal vector $\vec{n} = (a,b,c)$ is

$$\boxed{ax + by + cz + d = 0}$$

where $d = \underbrace{-(ax_0 + by_0 + cz_0)}$



Remark: d is independent of choices of P_0.

All points P_0 in the plane give the same d.

Remark. The linear equation of a plane is not unique.

- the normal vector (a,b,c) is determined only up to a nonzero constant.
- Once the normal vector $\vec{n}$ is chosen, d is independent of P_0.

The set of all planes in $\mathbb{R}^3$ is a 3-dimensional object (manifold).

$\downarrow$

$4-1$

Example Find an equation of the plane that passes through the points $P(1,3,2)$, $Q(3,-1,6)$, $R(5,2,0)$.

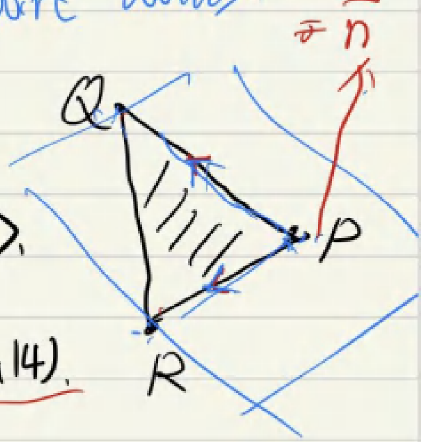
To find an equation, we need to find a point and a normal vector.

① • Point : $P(1,3,2)$. Q, R any point works.

② • normal vector :

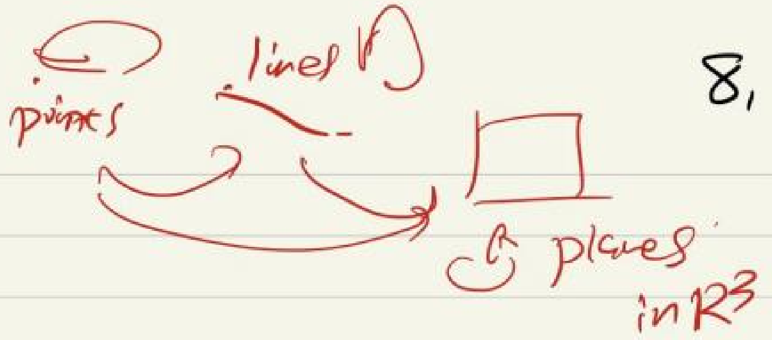
$$\vec{PQ} = \langle 2,-4,4\rangle, \qquad \vec{PR} = \langle 4,-1,-2\rangle.$$

$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i}, & \vec{j}, & \vec{k} \\ 2 & -4 & 4 \\ 4 & -1 & 2 \end{vmatrix} = \langle 12, 20, 14\rangle.$$



$$\Rightarrow \quad \boxed{12(x-1) + 20(y-3) + 14(z-2) = 0.}$$

$$6x + 10y + 7z - 50 = 0$$



3. Distances,

(1) distance between two points,

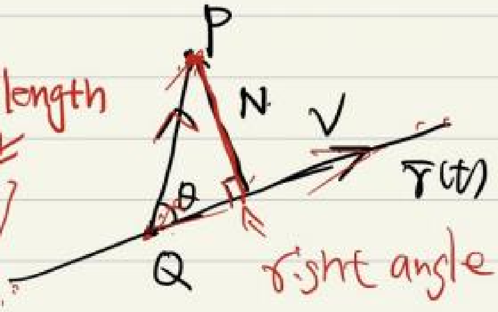
$P_1(x_1,y_1,z_1)$ $P_2(x_2,y_2,z_2)$,

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$ ✓

(2) distance between a point P to a line, $L: \vec{r}=\underline{Q}+t\vec{v}$

$$\text{Dist} = |N| = |\vec{QP}|\sin\theta$$
$$= \frac{|\vec{QP}||\vec{v}|\sin\theta}{|\vec{v}|} = \boxed{\frac{|\vec{QP}\times\vec{v}|}{|\vec{v}|}}$$

length



Example $P=(4,2,-1)$. $\vec{r}(t) = \underbrace{(1,2,-1)}_{Q} + t\underbrace{\langle 2,-1,3\rangle}_{\vec{v}}$.

(1) any Q. works

(2) any $\vec{v}$ works

$$\vec{QP} = (3,0,0) \quad \vec{v}=(2,-1,3)$$

$$\vec{QP}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 0 & 0 \\ 2 & -1 & 3 \end{vmatrix} = -9\vec{j}-3\vec{k}.$$

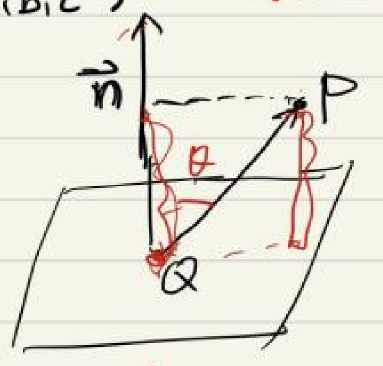
$$|\vec{QP}\times\vec{v}| = \sqrt{(-9)^2+(-3)^2} = \sqrt{90} = 3\sqrt{10}.$$

$$|\vec{v}| = \sqrt{2^2+(-1)^2+3^2} = \sqrt{14}.$$

$$\text{Dist} = \frac{3\sqrt{10}}{\sqrt{14}} = \frac{3\sqrt{140}}{14} = \frac{3\sqrt{35}}{7}.$$

(3) Distance between a point P to a plane $ax+by+cz+d=0$

Let Q be a point on the plane, $\vec{n}=(a,b,c)$ ← normal vecer



$$Dist = |Comp_{\vec{n}}\vec{QP}|$$

$$= \frac{|\vec{n}\cdot\vec{QP}|}{|\vec{n}|}$$ ← absolute value

$(\vec{n}\cdot\vec{QP})$

$= |\vec{n}|\cdot|\vec{QP}|\cdot\cos\theta$.

Example: $P=(1,-1,2)$. $2x+y-z-5=0$.

choose a point $Q=(0,5,0)$. ⇐ Q

$\vec{QP}=(1,-6,2)$.

$\vec{n}\cdot\vec{QP}=(2,1,-1)\cdot(1,-6,2)=-6$.

$|\vec{n}|=\sqrt{2^2+1^2+(-1)^2}=\sqrt{6}$.

$Dist = \frac{|-6|}{\sqrt{6}}=\sqrt{6}$.

Theorem If $P=(x_1,y_1,z_1)$, the plane is: $ax+by+cz+d=0$.

$$\boxed{Dist = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}}$$

Proof choose any point $(x_0,y_0,z_0)=Q$ in the plane.

$\vec{QP}=(x_1-x_0,y_1-y_0,z_1-z_0)$,

$\vec{n}\cdot\vec{QP} = (a,b,c)(x_1-x_0,y_1-y_0,z_1-z_0)$

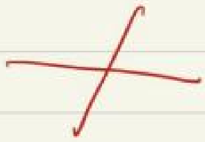
$= ax_1+by_1+cz_1-(ax_0+by_0+cz_0) = ax_1+by_1+cz_1+d$.

□

(4) Distance between two (different) lines. L_1 & L_2.

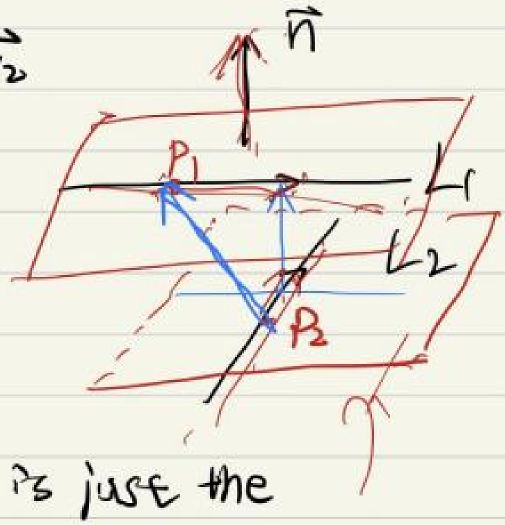
- parallel : choose any point P on L_1, the find the distance between P and L_2

- intersect.



- skew (NOT in the same plane).

L_1 : $P_1 + t v_1$ $\qquad$ L_2 : $P_2 + t\vec{v_2}$



Since lines are skew, they lie on parallel planes with normal vector.

$\vec{n} = \vec{v_1} \times \vec{v_2}$

The distance between these lines is just the distance between the two planes.

$$\text{Dist} = \left|\text{comp}_{\vec{n}} \overrightarrow{P_1P_2}\right| = \frac{|\vec{n} \cdot \overrightarrow{P_1P_2}|}{|\vec{n}|}$$

Example : $\vec{r}_1(t) = (1,2,-1) + t(-2,1,3)$
$\vec{r}_2(t) = (2,2,-1) + t(0,2,-3)$.

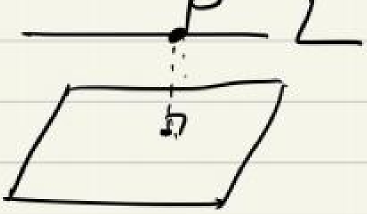
$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 1 & 3 \\ 0 & 2 & -3 \end{vmatrix} = -9\vec{i} - 6\vec{j} - 4\vec{k}$$

$\overrightarrow{P_1P_2} = (1,0,0)$, $\quad \overrightarrow{P_1P_2} \cdot \vec{n} = -9$, $\quad |\vec{n}| = \sqrt{9^2+4^2+6^2} = \sqrt{133}$

$$\text{Dist} = \frac{9}{\sqrt{133}}$$

(5) Distance between a line and a plane.

- L is in the plane.
- $L \cap$(the plane) = one point.
- L paraller to the plane

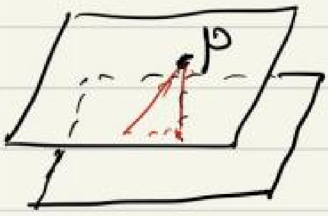


Find a point $P \in L$,

compute the distance between P and the plane.

(6). Distance between. two planes.

- they have nonempty intersections.
- they are parallel

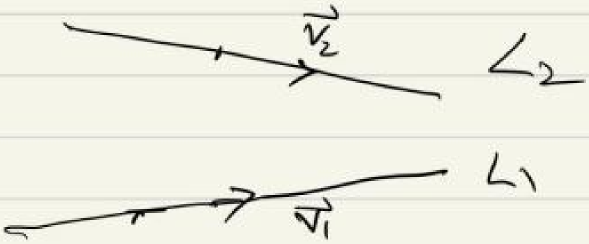


choose any point P on a Plane,

compute the distance between P and another plane,

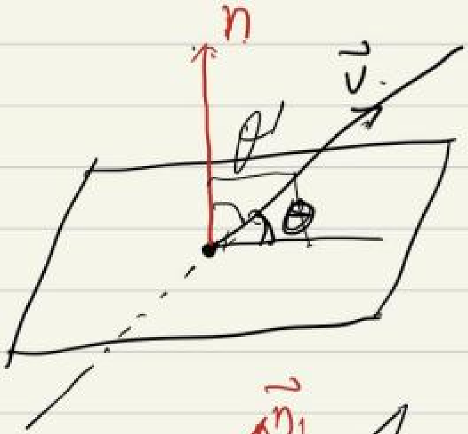
4. Angles, $(0 \le \theta \le \frac{\pi}{2})$.

(1) line to line.



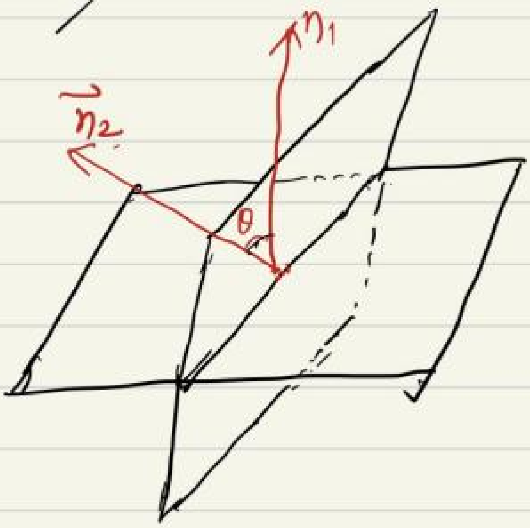
$$\underline{|\cos\theta|} = \frac{|\vec{v_1}\cdot\vec{v_2}|}{|\vec{v_1}|\,|\vec{v_2}|}$$

(2) line to plane



$$\sin\theta = \frac{|\vec{v}\cdot\vec{n}|}{|\vec{v}|\,|\vec{n}|}.$$

(3) plane to plane.



$$\underbrace{|\cos\theta| = \frac{|\vec{n_1}\cdot\vec{n_2}|}{|\vec{n_1}|\cdot|\vec{n_2}|}}$$

angle normal vector

||

angle planes.

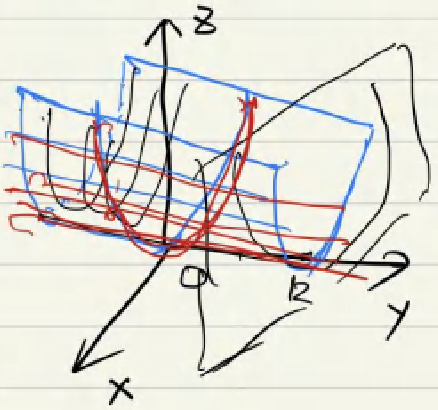
Lecture 4, Quadric surfaces, curves.

planes are simplest surfaces: a plane is just the zero locus of a linear equation. we investigate two other types of surfaces: cylinders and quadric surfaces.

$Ay+By+Cz+d=0$

1. Cylinders.

A cylinder is a surface that consists of all lines (called rulings) that are parallel to a given line and pass through a given plane curve.

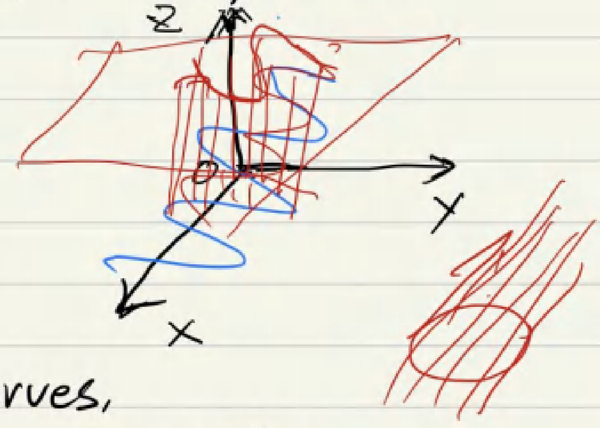


Example. $z=x^2 \Leftarrow y$

- any vertical plane $y=k$ intersects the graph in a curve with equation $z=x^2$. So these vertical traces are parabolas

- The graph is a surface, called parabolic cylinder.

Example $y=\sin x \cdots$ ⇐ transendental.



Cylinders are "induced" from curves.
Geometrically, we can write a cylinder S as

$$S=C\times L,$$

where C is a plane curve and L is a line.

algebraic geometry

deg 1 ↖ line planes ← algebraic surfaces → polynomials

2,

2, Quadric surfaces.

A quadric surface is the graph of a degree-2 polynomial in three variables x, y and z. The most general such equation is

$$\underbrace{Ax^2+By^2+Cz^2+Dxy+Eyz+Fxz}+Gx+Hy+Iz+J=0$$

where A, B, C ⋯ J are constants,

$(x-1)^2+y^2+z^2=1 \Leftrightarrow x^2+y^2+z^2=1$

We need to simplify this equation by some good transformations (those do not change the shape)

Step 1. By linear algebra, the degree 2 parts can be brought into standard forms (by rotations, or reflections).

$Ax^2+By^2+Cz^2$ ← diagonal forms

or Ax^2+By^2

or Ax^2.

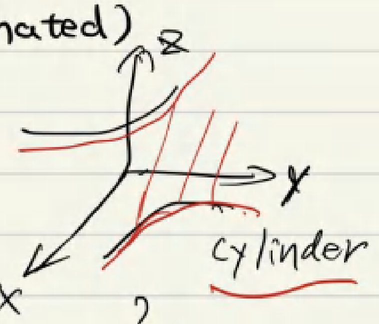
(i.e, those xy, yz, xz can be eliminated)

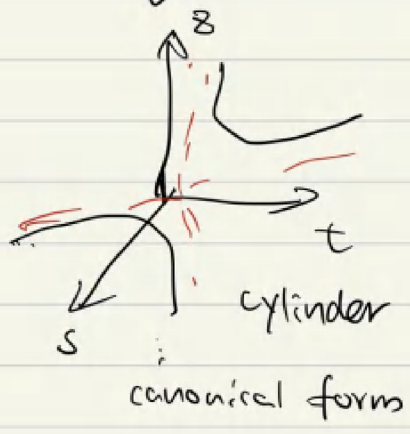
Example $xy-1=0$

take. $x=s+t \quad y=s-t$

$\Rightarrow \boxed{s^2-t^2=1.} \Leftarrow s^2-t^2=1$

This linear transformation is NOT unitary, but a constant multiple is.





canonical form

$x^2+y^2+z^2+\boxed{ax}+by+cz+d=0$

$(x-\frac{a}{2})^2 - (\quad)+$
$+d'=0$

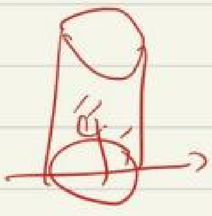
Step 2 • If the degree 2 part is

$$Ax^2+By^2+Cz^2.$$ $(A, B, C \neq 0)$

we can alway complete squares to eliminate degree 1 terms.
Geometrically, we are translating the surface.

So we get

① $\boxed{Ax^2+By^2+Cz^2+J=0}$



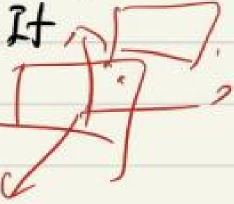
• If the degree 2 part is
Ax^2+By^2, $A, B \neq 0.$

$x^2+y^2-1=0$

we can eliminate the terms Gx, Hy,
If $I=0$. $Ax^2+By^2+J=0$ is a cylinder.
If $I\neq 0$, by a translation (in z), we get

② $\boxed{Ax^2+By^2+Iz=0}$

• If the degree 2 part is Ax^2,
we can eliminate Gx. If $Hy+Iz\neq 0$, by a change of coordinate, we get. $Ax^2+y=0$. A cylinder. If $Hy+Iz=0$, $Ax^2=J$. union of planes.



Conclusion We only need to discuss two classes of quadric surfaces.

3. Six fundamental quadric surfaces.

$x^2+y^2+z^2=0$
$\rightarrow (0,0,0)$

Case (i)

$$Ax^2+By^2+Cz^2+J=0.$$

$Ax^2+By^2+Cz^2=0.$

- If $J=0$, A, B, C cannot all be positive or negative

  we assume $A, B>0$, $C<0$. we get.

$\begin{cases} Ax^2+By^2+Cz^2= \\ -(Ax^2+By^2+Cz^2)=0 \end{cases}$

cone: $\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ ①

$f \Updownarrow cf$ $c\neq0$. define the same surface

- If $J\neq0$, we may assume $J=1$

  If $A, B, C>0$, we get

ellipsoid. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ②

If two of $A, B, C>0$. we get

hyperboloid of one sheet
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ ③

If one of $A, B, C>0$, we get

hyperboloid of two sheets.
$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ④

If $A, B, C < 0$. empty set, $\quad -x^2-y^2-z^2=1$

ϕ

Case (ii),

$$\underline{Ax^2+By^2}+\underline{I}z=0,$$

If $A \cdot B > 0$,

elliptic paraboloid.

$$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

⑤

If $A \cdot B < 0$.

hyperbolic paraboloid.

$$\frac{z}{c}=\frac{y^2}{b^2}-\frac{x^2}{a^2}$$

(saddle). ⑥

Remark: To, determine the shape of a quadric surface,

1) Case (i) or case (ii),

2), signs of the coefficients.

↑

① how to "see" the type of quadric surface

② where is —

↑ look a place

4, Some pictures.

Example Sketch the quadric surface with equation.

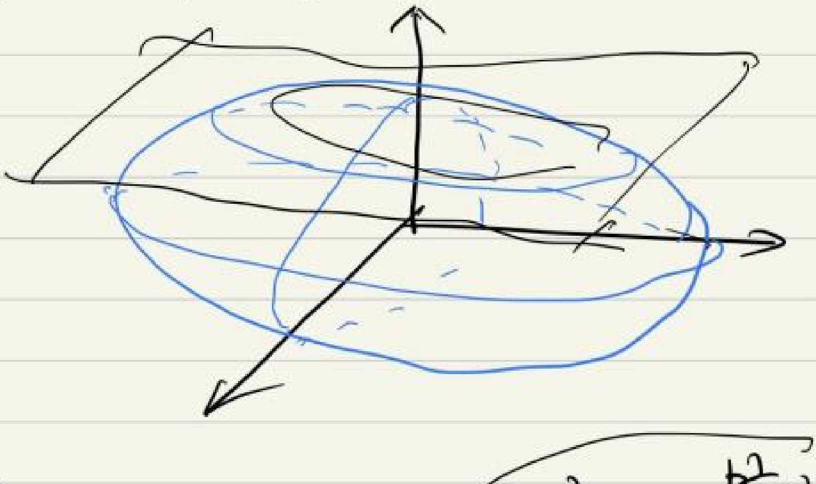
$$x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1. \qquad (z=k)$$

Solution. By substituting $z=0$. we find that the trace in the xy-plane is $x^2+y^2/9=1$, an ellipse,

In general, the horizontal trace in the plane $z=k$ is

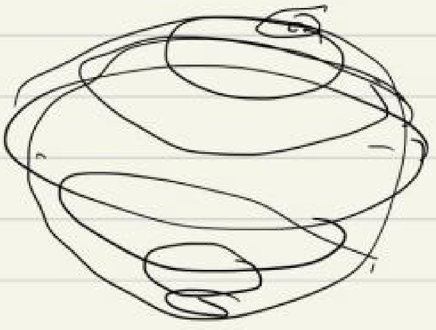
$$x^2 + \frac{y^2}{9} = 1 - \frac{k^2}{4}, \qquad z=k;$$

which is an ellipse, provided that $k^2<4$,



$$\begin{cases} x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1 \\ z=k. \end{cases} \rightsquigarrow x^2 + \frac{y^2}{9} = 1 - \frac{k^2}{4}$$

$-2 \cancel{0} \le k \le 2$



Example sketch the surface

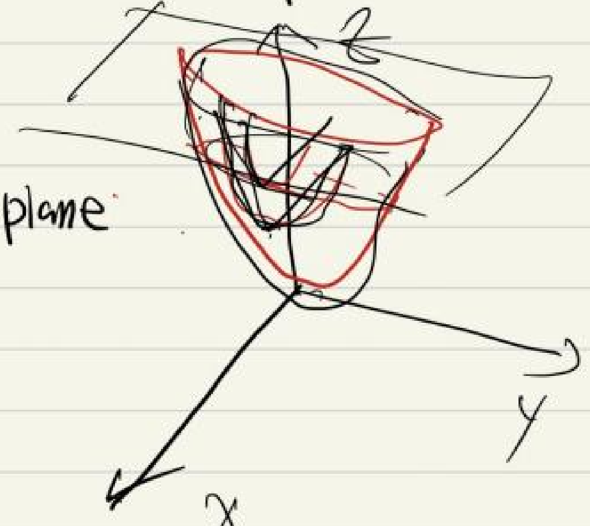
$z = 4x^2 + y^2$.

$x = k$. $z = y^2 + 4k^2$

Solution, ① If we put $x=0$, we get $y = z^2$, so the yz-plane intersects the surface in a parabola,

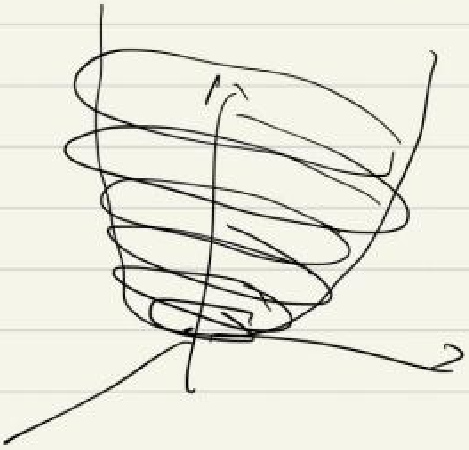
If we put $x=k$, we get $z = y^2 + 4k$,

This means that if we slice the graph with any plane parallel to the yz-plane, we obtain a parabola that opens upward,



② The horizontal trace in. the plane $z=k$ is an ellipse,
$(k>0)$

$4x^2 + y^2 = k$ → $k>0$



$$t \longrightarrow (x(t), y(t), z(t))$$

$$\mathbb{R}^m \longrightarrow \mathbb{R}^n$$

$m=1$

5. Vector functions.

A vector function, is simply a function whose domain is a set of real numbers, and whose range is a set of vectors.

If the vectors are three-dimensional, we may find. real functions $f(t), g(t), h(t)$, so that

$$\vec{r}(t) = (f(t), g(t), h(t)) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}.$$

f, g, h are called the component functions of $\vec{r}$.

$$\begin{tikzcd} \text{vector} \arrow[r, leftrightarrow] \arrow[d] & \text{triple} \arrow[d] \\ \underline{\text{vector function}} \arrow[r, leftrightarrow] & \text{ordered real functi} \end{tikzcd}$$

6. Limits and continuity

Definition If $\vec{r}(t) = (f(t), g(t), h(t))$, then.

$$\lim_{t\to a} \vec{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \rangle.$$

provided the limits of the component functions exist.

f(t g(t) h(t) real function

Example. $\vec{r}(t) = (1+t^3)\vec{i} + te^{-t}\vec{j} + \frac{\sin t}{t}\vec{k}$ $\mathbb{R} \to \mathbb{R}$

what is $\lim_{t\to 0} \vec{r}(t)$?

$e^{-t} \to 1.$

Solution: $\lim_{t\to 0} (1+t^3) = 1$, $\lim_{t\to 0} te^{-t} = 0$, $\lim_{t\to 0} \frac{\sin t}{t} = 1$. ✓

t=0 famous

$$\Rightarrow \lim_{t\to 0} \vec{r}(t) = \vec{i} + \vec{k} = (1, 0, 1)$$

A vector function is continuous at a if

$$\lim_{t\to a} \vec{r}(t) = \vec{r}(a).$$

$\vec{r}$ is continuous $\overset{\text{at } a}{\Longleftrightarrow}$ its component functions are continuous at a.

7, Derivatives.

Definition

$$\boxed{\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}} \leftarrow$$

Theorem. If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$, where f, g, h are differentiable functions, then,

$$\vec{r}\,'(t) = (\underline{f'(t)}, \underline{g'(t)}, \underline{h'(t)}).$$

Example. Find the derivative of

$$\vec{r}(t) = \underline{(1+t^3)}\vec{i} + \underline{te^{-t}}\,\vec{j} + \underline{\sin 2t}\,\vec{k}.$$

Solution

$$\vec{r}\,'(t) = 3t^2\vec{i} + (1-t)e^{-t}\,\vec{j} + 2\cos 2t\,\vec{k}.$$

Differential rules:

$\vec{u}, \vec{v}$, differentiable vector functions, c a scalar, f real function.

1. $\frac{d}{dt}(\vec{u}(t)+\vec{v}(t)) = \vec{u}'(t) + \vec{v}'(t)$, $\Big\}$ linear
2. $\frac{d}{dt}(c\vec{u}(t)) = c\,\vec{u}'(t)$,
3. $\frac{d}{dt}(f(t)\vec{u}(t)) = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$, ← Leibniz rule.
4. $\frac{d}{dt}(\vec{u}(t)\cdot\vec{v}(t)) = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)$, $\Big\}$ corollary of Leibniz rule.
5. $\frac{d}{dt}(\vec{u}(t)\times\vec{v}(t)) = \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t)$.
6. $\frac{d}{dt}(\vec{u}(f(t))) = f'(t)\,\vec{u}'(f(t))$, chain rule.

derivative of a function ←

vector function ←

Proof of 4:

Let $\vec{u}(t) = (f_1(t), f_2(t), f_3(t))$,

$\vec{v}(t) = (g_1(t), g_2(t), g_3(t))$

$$\vec{u}(t)\cdot\vec{v}(t) = \sum_i f_i(t)\,g_i(t),$$

$$\frac{d}{dt}(\vec{u}(t)\cdot\vec{v}(t)) = \frac{d}{dt}\Big(\sum_i f_i(t)\,g_i(t)\Big)$$

$\big\downarrow$ Leibniz

$$= \sum_i f_i'(t)\,g_i(t) + \sum_i f_i(t)\,g_i'(t).$$

$$= \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t), \qquad \square$$

Definition $$\int_a^b \vec{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\vec{i} + \left(\int_a^b g(t)\,dt\right)\vec{j} + \left(\int_a^b h(t)\,dt\right)\vec{k}.$$

Example. $\vec{r}(t) = 2\cos t\,\vec{i} + \sin t\,\vec{j} + 2t\,\vec{k}$

$$\int \vec{r}(t)\,dt = 2\sin t\,\vec{i} - \cos t\,\vec{j} + t^2\vec{k} + \vec{C}$$

$$\int_0^{\frac{\pi}{2}} \vec{r}(t)\,dt = \left(2\sin t\,\vec{i} - \cos t\,\vec{j} + t^2\vec{k}\right)\Big|_0^{\frac{\pi}{2}}$$

$$= 2\vec{i} + \vec{j} + \frac{\pi^2}{4}\vec{k}.$$

vector function ⟷ ordered pair of 3 <u>real</u> function.

limit continuity, derivative, integral.

↓

componentwise

□

Lecture Notes. Quadric Surfaces.

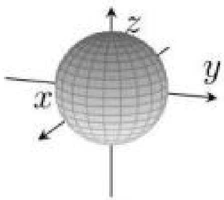
These are analogous to conic sections in $\mathbb{R}^2$. Quadric Surfaces are defined by a quadratic equation in x, y and z.

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx + Gx + Hy + Iz + J = 0$$

There are six fundamental quadric surfaces.

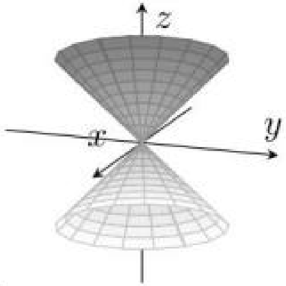
Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



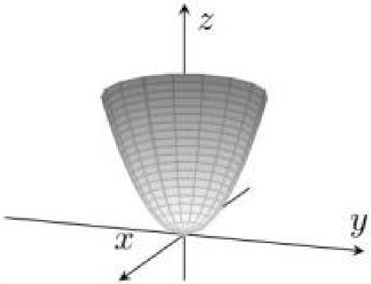
Cone

$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$



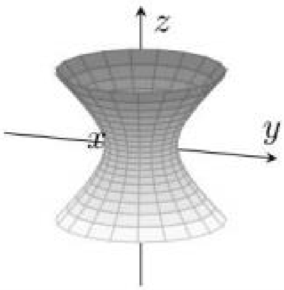
Elliptic Paraboloid

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$



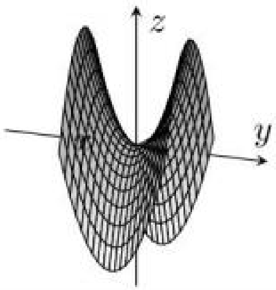
Hyperboloid of One Sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$



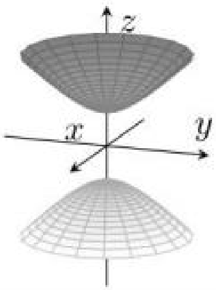
Hyperbolic Paraboloid (Saddle)

$$\frac{z}{c} = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$



Hyperboloid of Two Sheets

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



Lecture 5. Space curves, arc length, and curvature

vector function → (f, g, h)

1. Space curves.

Suppose that f, g, and h are continuous real-valued functions on an interval I. Then the set C of all points (x, y, z) in space, where

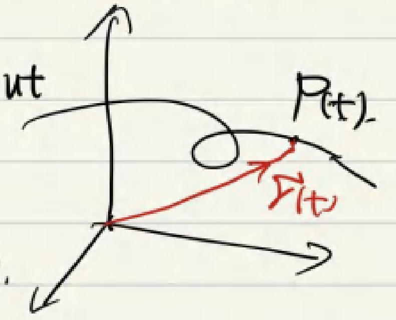
$$x = f(t), \quad y = g(t), \quad z = h(t),$$

and t varies throughout the interval I, is called a space curve. The equations are called parametric equations of C and t is called a parameter.

↑ NOT unique.

t ↗
~~time.~~
t
↓
(x, y, z)
$= (f(t), g(t), h(t))$

If we consider the vector function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, then $\vec{r}(t)$ is the position vector of the point $P(f(t), g(t), h(t))$. Thus any continuous vector function $\vec{r}$ defines a space curve that is traced out by the tip of the moving vector $\vec{r}(t)$.



$$\underset{\text{(function)}}{\text{vector function}} \longrightarrow \underset{\text{(set)}}{\text{space curve.}}$$

intrinsically, does NOT depend on parametric equations

Parametric functions ⟶ curves,

Example. Sketch the curve whose vector equation is

$$\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\vec{k}.$$

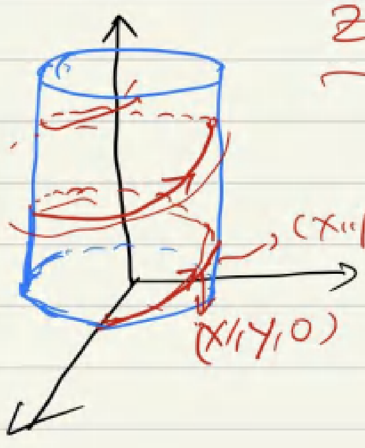
$(\cos t, \sin t, 0)$

2,

$z = v$

Solution. The parametric equations for this curve are

$x = \cos t \quad y = \sin t \quad z = t$



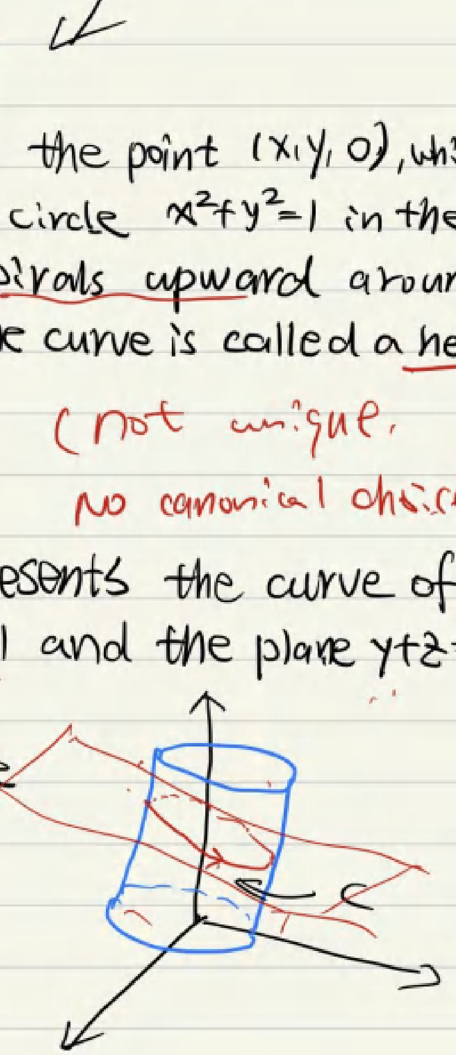
· The curve must lie on the circular cylinder $x^2 + y^2 = 1$. $\cos^2 t + \sin^2 t = 1$

· The point (x, y, z) lies directly above the point $(x, y, 0)$, which moves counterclockwise around the circle $x^2+y^2=1$ in the xy-plane. Since $z=t$, the curve spirals upward around the cylinder as t increases. The curve is called a helix

curve ⟶ parametric functions (not unique, No canonical choices)

Example Find a vector function that represents the curve of intersection of the cylinder $x^2+y^2=1$ and the plane $y+z=2$

· The projection of C onto the xy-plane is the curve $x^2+y^2=1$, $z=0$.



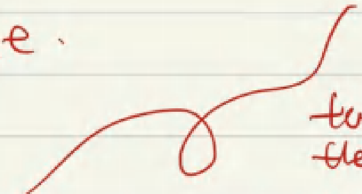
$x = \cos t \quad y = \sin t, \quad 0 \le t \le 2\pi.$

then $z = 2 - y = 2 - \sin t$

So the corresponding vector function is

$$\vec{r}(t) = \cos t\, \vec{i} + \sin t\, \vec{j} + (2 - \sin t)\vec{k}, \quad 0 \le t \le 2\pi.$$

canonical parameter for $C = S^1$, circle.



vector functions → curve

vector set

take the image

ignore the map.

2. Tangent vector

$\mathbb{R} \to \mathbb{R}$

f

derivative functions.

$f(x) = c$

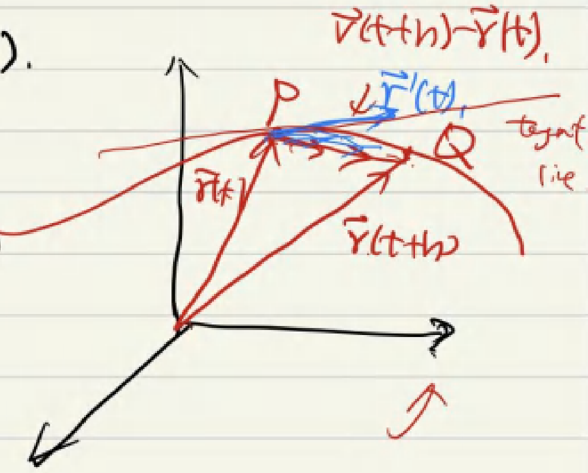
(arbitrary)

Let $\vec{r}$ be a vector function,

$$\boxed{\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}}$$

In coordinates, if $\vec{r}(t) = (f(t), g(t), h(t))$,

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle.$$



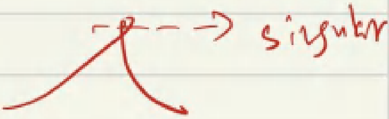
If the points P and Q have position vectors $\vec{r}(t)$ and $\vec{r}(t+h)$,

$\vec{r}(t+h) - \vec{r}(t) = \vec{PQ}$

represents a secant vector.

As $h \to 0$, $(1/h)(\vec{r}(t+h) - \vec{r}(t))$ approaches a vector that lies on the tangent line.

The vector $\vec{r}\,'(t)$ is called the tangent vector to the curve defined by $\vec{r}$ at point P, provided that $\vec{r}(t)$ exists and $\boxed{\vec{r}\,'(t) \neq \vec{0}. ☆}$



The tangent line to C at P is defined to be the line through P parallel to the tangent vector $\vec{r}\,'(t)$.

The tangent unit vector is $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ (if $\vec{r}\,'(t) \neq \vec{0}$)

Remark: Different vector functions may represent the same curve, To discuss tangent vectors, we must first fix parametric equation.

$(x=x_0+at,\ y=y_0+bt,\ z=z_0+ct))$ represents a line, the tangent vector at any point P is (a,b,c).

If we replace (a,b,c) with $(\lambda a, \lambda b, \lambda c)$, we get the same line, but different tangent vectors,

However, tangent lines are independent of the choices of parametric equations. (directions of tangents is independent of choices of parametric equations),

3. Arc length.

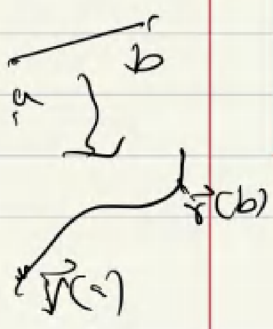
The length of a plane curve with parametric equations $x=f(t)$, $y=g(t)$, $a\le t\le b$, is defined as the limit of lengths of inscribed polygons, For ~~the~~ case where f' and g' are continuous, we arrived at the formula.

$$L=\int_a^b\sqrt{(f'(t))^2+(g'(t))^2}\,dt. \quad ☆$$

$a\le t\le b.$

Now suppose, the curve has the parametric equations. $x=f(t)$, $y=g(t)$, $z=h(t)$, where f', g', and h' are continuous. If the curve is tranversed exactly once as t increases, from a to b, its length is

$$\boxed{L=\int_a^b\sqrt{(f'(t))^2+(g'(t))^2+(h'(t))^2}\,dt=\int_a^b|\vec{r}'(t)|\,dt}$$



Example Find the length of the arc of the circular helix with vector equation

$$\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\vec{k}$$

from the point $(1,0,0)$ to the point $(1,0,2\pi)$.

$t=0$ $t=2\pi$

Solution $\underline{\vec{r}'(t)} = -\sin t\,\vec{i} + \cos t\,\vec{j} + \vec{k}.$

$$|\vec{r}'(t)| = \sqrt{(-\sin t)^2+(\cos t)^2+1} = \sqrt{2}.$$

$$L = \int_0^{2\pi} |\vec{r}'(t)|\,dt = 2\sqrt{2}\pi.$$

$$\int \sqrt{\boxed{(\quad)^2}}\,dt$$

very complicated function of t

Remark A single curve C can be represented by more than one vector functions. However, the arc length is independent of the parameterization that is used. The proof is "change of variable". So the arc length is intrinsically defined

geometric quantity.

$$\int_a^b (\quad)\,dt$$

$$\shortparallel$$

$$\int_c^d (\quad)\,ds$$

We define the arc length function s of a curve represented by $\vec{r}(t)$, by $a \le t \le b$.

$$\boxed{s(t) = \int_a^t |\vec{r}'(u)|\,du.}$$

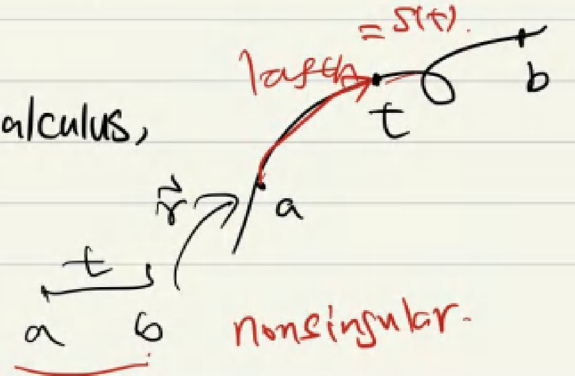
That is, $s(t)$ is the length of the part of C between $\vec{r}(a)$ and $\vec{r}(t)$.

By fundamental theorem of calculus,

$$\frac{ds}{dt} = \underline{|\vec{r}'(t)|} \geq 0.$$

> 0

if curve is



nonsingular.

It is often useful to parametrize a curve with respect to arc length, because arc length arises naturally from the shape of the curve and does not depend on a particular coordinate system.

→ canonical.

Example. $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\vec{k}$, $a = 0$.

$|\vec{r}'(t)| = \sqrt{2}$

Solution: $s = \sqrt{2}t. \Rightarrow t = s/\sqrt{2}$

$$\vec{r}(t(s)) = \cos(s/\sqrt{2})\,\vec{i} + \sin(s/\sqrt{2})\,\vec{j} + s/\sqrt{2}\,\vec{k},$$

$s \to t \to \vec{r}$

$s \rightleftarrows t$

- disadvantage.
  hard to compute
  practically not s-good

- advantage
  proof theorems
  theoretically useful

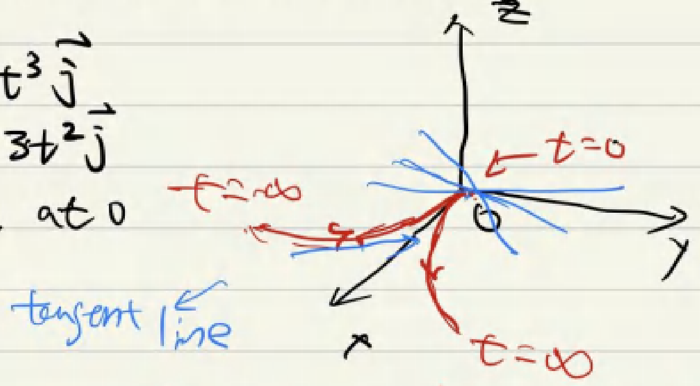
4 Curvature.

A parametrization $\vec{r}(t)$ is called smooth on an interval I if $\vec{r}'$ is continuous and $\vec{r}'(t) \neq \vec{0}$ on I. A curve is called smooth if it has a smooth parametrization.

A smooth curve has no sharp corners or cusps; when the tangent vector turns, it does so continuously.

Example.

$$\vec{r}(t) = t^2\,\vec{i} + t^3\,\vec{j}$$
$$\vec{r}'(t) = 2t\,\vec{i} + 3t^2\,\vec{j}$$
$$\vec{r}'(0) = \vec{0} \Rightarrow \text{cusp. at } 0$$



Cannot find tangent vector

at $\vec{r}(0) = (0,0,0)$

If C is a smooth curve defined by the vector function $\vec{r}$, the unit tangent vector $\vec{T}(t)$ given by.

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

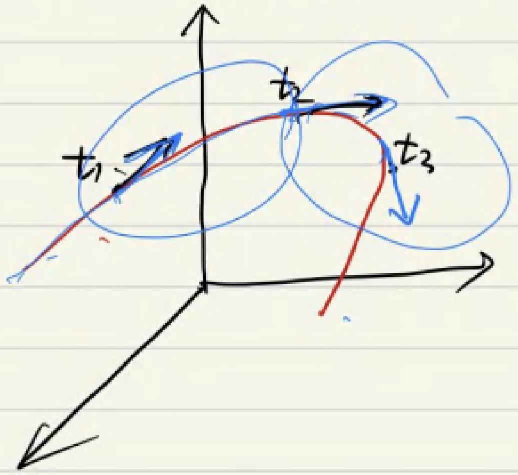
is defined everywhere.

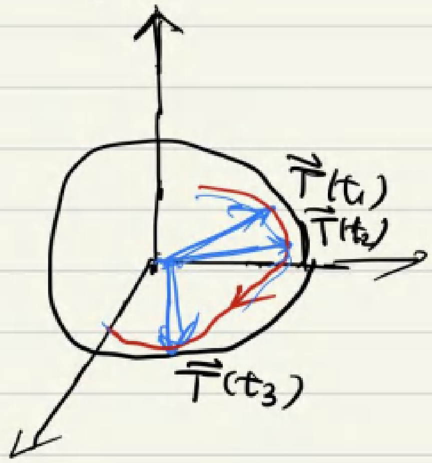
$(\|\vec{T}(t)\| = 1$

$t \to \vec{T}(t)$

$\vec{T}(t)$ is a function : $\mathbb{R} \to S^2$. (the Gauss map).

$\vec{T}(t)$ changes direction very slowly when C is fairly straight, but it changes direction more quickly when C bends or twists more sharply.





The curvature of C at a given point is a measure of how quickly the curve changes direction at that point. Specifically, we define it to be the magnitute of the rate. of change of the unit tangent vector with respect to arc length:

s : parame- arc-length.

Definition. The curvature of a curve is

$$\boxed{\kappa = \left|\frac{d\vec{T}}{ds}\right|}$$

$s \to t \to T$

where $\vec{T}$ is the unit tangent vector.

tangent NOT → directions are the same orientation

✓

Remark: $\vec{T}$ and s are both defined intrinsically, so is κ. (independent of the parametrization).

Usually it is not easy to compute $s = s(t)$ explicitly,

Fortunately, we can express the curvature in terms of the parameter t instead of s,

$$\boxed{\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}} \quad ☆$$

$$\left(T = \frac{v}{\text{[illegible]}}\right)'$$

messy

Proof: the chain rule:

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds} \cdot \frac{ds}{dt}$$

$$\Rightarrow \kappa = \frac{d\vec{T}}{ds} = \frac{|d\vec{T}/dt|}{|ds/dt|} \rightarrow |\vec{r}'(t)|$$

Example Show that the curvature of a circle of radius a is $1/a$.

Solution: $\vec{r}(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j}$.

$$\vec{r}'(t) = -a\sin t\,\vec{i} + a\cos t\,\vec{j} \qquad |\vec{r}'(t)| = a.$$

$$\vec{T}(t) = \frac{\vec{r}(t)}{|\vec{r}'(t)|} = -\sin t\,\vec{i} + \cos t\,\vec{j}$$

$$\vec{T}'(t) = -\cos t\,\vec{i} - \sin t\,\vec{j}. \qquad |\vec{T}'(t)| = 1$$

$$\Rightarrow \kappa = \frac{1}{a}.$$

radius a → $\underline{\text{curvature } \frac{1}{a}}$.

a constant

(2)
Small circles have large curvature and large circles have small curvature.

There is a more convenient formula of κ.

Theorem The curvature of the curve given by the vector function $\vec{r}$ is

$$\boxed{\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|}}$$

(1) Circle looks "the same" at all points

Proof Since $\vec{T} = \frac{\vec{r}'}{|\vec{r}'|}$ and $|\vec{r}'| = ds/dt$, we have.

$$\vec{r}' = |\vec{r}'|\vec{T} = \frac{ds}{dt}\vec{T}.$$

no functions in denominators

$$\Rightarrow \vec{r}'' = \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\vec{T}'.$$

$$\vec{r} \times \vec{r}'' = \left(\frac{ds}{dt}\right)^2 \vec{T} \times \vec{T}' \qquad (\vec{T} \times \vec{T} = 0).$$

Now $|\vec{T}(t)| = 1 \Rightarrow \vec{T}(t) \cdot \vec{T}(t) = 1.$

$$\frac{d}{dt}(\vec{T}(t) \cdot \vec{T}(t)) = 2\vec{T}(t) \cdot \vec{T}'(t) = 0$$

$\Rightarrow \vec{T}'(t)$ is orthogonal to $\vec{T}(t)$.

$$|\vec{r} \times \vec{r}''| = \left(\frac{ds}{dt}\right)^2 |\vec{T}||\vec{T}'| = \left(\frac{ds}{dt}\right)^2 |\vec{T}'|.$$

$$\Rightarrow |\vec{T}'| = \frac{|\vec{r}' \times \vec{r}''|}{|ds/dt|^2} = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^2}$$

$$\kappa = \frac{|\vec{T}'|}{|\vec{r}|} = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3}$$

Example Find the curvature of the twisted cubic

$$\vec{r}(t) = t\vec{i} + t^2\vec{j} + t^3\vec{k}.$$

Solution. $\vec{r}\,'(t) = (1, 2t, 3t^2)$, $\vec{r}\,''(t) = (0, 2, 6t)$,

$$\vec{r}\,' \times \vec{r}\,''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = (6t^2, -6t, 2).$$

$$|\vec{r}\,' \times r''| = \sqrt{36t^4 + 36t^2 + 4} = 2\sqrt{9t^4 + 9t^2 + 1}$$

$$|\vec{r}\,'(t)| = \sqrt{9t^4 + 4t^2 + 1}$$

$$\kappa(t) = \frac{2\sqrt{1+9t^2+9t^4}}{(1+4t+9t^4)^{3/2}}.$$

At the origin, where $t=0$, $\kappa(0) = 2$.

For the special case of a plane curve with equation $y = f(x)$, we choose x as the parameter:

$$\vec{r}(x) = x\vec{i} + f(x)\vec{j}. \; (+ (0)\vec{k})$$

$\vec{r}\,'(x) = \vec{i} + f'(x)\vec{j}$. $\vec{r}\,''(x) = f''(x)\vec{j}$

$\vec{r}\,' \times \vec{r}\,'' = f''(x)\vec{k}$. $|\vec{r}\,'(x)| = \sqrt{1 + f'(x)^2}$.

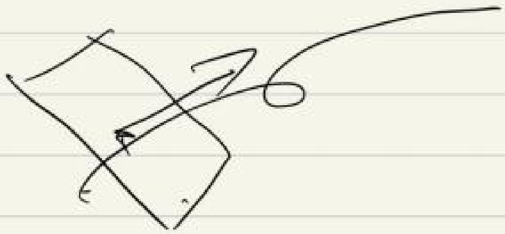
$$\Rightarrow \quad \boxed{\kappa(x) = \frac{|f''(x)|}{\sqrt{1 + (f'(x))^2}}.}$$

consider C
as a space curve
$\mathbb{R}^3$, $\vec{a} \times \vec{b}$.
is defined

Example Find the curvature of the parabola $y = x^2$.

Solution $y' = 2x \quad y'' = 2$,

$$\kappa = \frac{2}{(1+4x^2)^{3/2}}$$



5. The normal and binormal vectors.

At a given point on a smooth space curve $\vec{r}(t)$, there are many vectors that are orthogonal to the unit tangent vector $\vec{T}(t)$, ⟶ plane.

Sine $|\vec{T}(t)| = 1$, $\vec{T}'(t) \cdot \vec{T}(t) = 0$. At any point where $\kappa \neq 0$, we can define the (principal) unit normal vector $\vec{N}(t)$.

$$\boxed{\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}}$$

We can think of the unit normal vector as indicating the direction in which the curve is turning at each point.

The vector

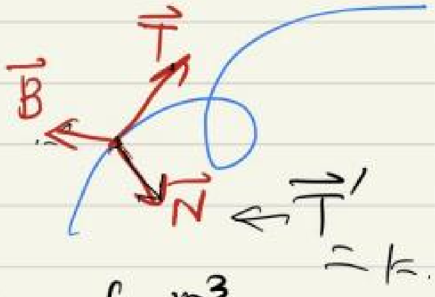
$$\boxed{\vec{B}(t) = T(t) \times \vec{N}(t)}$$

is called the binormal vector. It is perpendicular to both $\vec{T}$ and $\vec{N}$ and is also a unit vector.

In conclusion, at each point $\kappa \neq 0$.
we have $\{\vec{T}, \vec{N}, \vec{B}\}$.

- unit vectors,
- orthogonal pairwise, $\Big\}$

$\Rightarrow$ a frame of $\mathbb{R}^3$.
$(\vec{i}, \vec{j}, \vec{k})$



moving frame

Remark: In arc-length coordinate: (Frénet-Serret equations).

$$\frac{d}{ds}\begin{pmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\end{pmatrix} = \begin{pmatrix} & \kappa & \\ -\kappa & & \tau \\ & -\tau & \end{pmatrix}\begin{pmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\end{pmatrix}$$

differential geometry

κ: curvature, τ: torsion.

It turns out that real function $\kappa(s)$ and $\tau(s)$ determine the shape of space curves.

two real quantity
$\downarrow$
space curve

Example Find $\vec{T}, \vec{N}, \vec{B}$ for the circular helix
$$\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\vec{k}.$$

Solution. $\vec{r}'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j} + \vec{k}$. $\quad |r'(t)| = \sqrt{2}$

$\vec{T}(t) = \frac{1}{\sqrt{2}}(-\sin t\,\vec{i} + \cos t\,\vec{j} + \vec{k})$.

$\vec{T}'(t) = \frac{1}{\sqrt{2}}(-\cos t\,\vec{i} - \sin t\,\vec{j}) \quad \vec{T}'(t) = \frac{1}{\sqrt{2}}$,

$\vec{N}(t) = (-\cos t, -\sin t, 0)$.

$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \frac{1}{\sqrt{2}}(\sin t, -\cos t, 1)$. $\square$

Lecture 6,

1, Review.

vector function. $\longrightarrow$ curve

$\vec{r}(t). = (x(t), y(t), z(t)),$ the set of points.

$a \leq t \leq b.$ $(x(t).\ y(t),\ z(t)),$

$\mathbb{R} \longrightarrow \mathbb{R}^3.$ function

we are interested in intrinsically defined quantities, i.e, those do not depend on the choices of parameterizations.

- arc length,

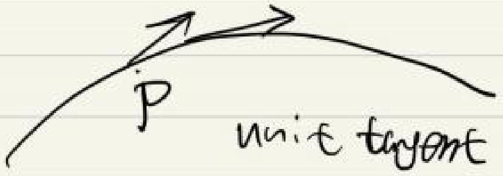
$$L(C) = \int_a^b |\vec{r}'(t)|\, dt.$$

- curvature:

a measure of how the curve changes direction at a point.

arc length parameter:

$$\kappa = \left|\frac{d\vec{T}}{ds}\right|$$



unit tangent vectors $\vec{T}$

$\mathbb{R} \to S^2$

general parameter:

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

theoretically useful

useful for explicit computations

The normal and binormal vector.

(defined at smooth points, $\kappa(t) \neq 0$).

$\vec{r}'(t) \neq \vec{0}$

$\vec{T}'(t) \neq 0$

a line. ↑ $\kappa = 0$

$\vec{r}(t) \longrightarrow \vec{T}(t)$ unit tangent vector

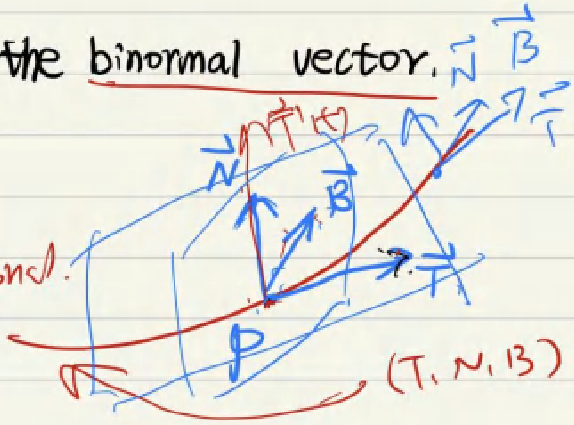
$|\vec{T}(t)| = 1 \Rightarrow \vec{T}(t) \cdot \vec{T}'(t) = 0.$ ←

$\frac{d}{dt} |\vec{T}(t)|^2 = 2\, \vec{T}(t) \cdot \vec{T}'(t) = 0$

the (principal) unit normal vector.

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}.$$

$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ is called the binormal vector.



$(\vec{i}, \vec{j}, \vec{k})$ ⇕

- $|\vec{T}| = |\vec{N}| = |\vec{B}| = 1.$ unit
  $\vec{T} \cdot \vec{N} = \vec{N} \cdot \vec{B} = \vec{T} \cdot \vec{B} = 0,$ orthogonal.
- $\vec{T} \times \vec{N} = \vec{B} \quad \vec{N} \times \vec{B} = \vec{T}, \quad \vec{B} \times \vec{T} = \vec{N}.$

In conclusion, at each point $P = (x(t), y(t), z(t))$, $\{\vec{T}, \vec{N}, \vec{B}\}$ is a (good) frame for $\mathbb{R}^3$.

As t varies, we get a family of frames.

The plane determined by the normal and binormal vectors $\vec{N}$ and $\vec{B}$ at a point on a curve C is call the normal plane of C at P. It consists of all lines that are orthogonal to the tangent vector $\vec{T}$.

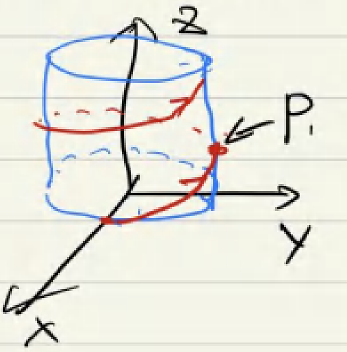
unit

↑ if $\kappa(t) = 0$

we can still define normal plane

The plane determined by the vectors $\vec{T}$ and $\vec{N}$ is called the osculating plane of C at P. It is the plane that comes closest to containing the part of the curve near P.

Example $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\vec{k}$ (circular helix)



← tangent vector

(1). $\vec{r}'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j} + \vec{k}.$

$|r'(t)| = \sqrt{2},\ = \sqrt{(-\sin t)^2 + \cos^2 t + 1}$

① $\Rightarrow \vec{T}(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{1}{\sqrt{2}}(-\sin t\,\vec{i} + \cos t\,\vec{j} + \vec{k}).$

(2). $\vec{T}'(t) = \frac{1}{\sqrt{2}}(-\cos t\,\vec{i} - \sin t\,\vec{j})$ $\quad |\vec{T}'(t)| = \frac{1}{\sqrt{2}}.$

② $\vec{N}(t) = \dfrac{\vec{T}'(t)}{|\vec{T}'(t)|} = (-\cos t\,\vec{i} - \sin t\,\vec{j})$

(3) $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \frac{1}{\sqrt{2}} \langle \sin t, -\cos t, 1 \rangle.$

③

(4) $P = (0, 1, \pi/2) = \vec{r}(\frac{\pi}{2})$ $\quad \vec{N}, \vec{B}$

the normal plane at P has normal vector.

$(\vec{T}, \vec{N}, \vec{B})$

$\vec{r}'(\frac{\pi}{2}) = \langle -1, 0, 1 \rangle.$

$\vec{T} = \vec{r}'/|r'|$

So an equation is

$-1(x-0) + 0(y-1) + 1(z - \frac{\pi}{2}) = 0 \Rightarrow z = x + \frac{\pi}{2}.$

(5). The osculating plane at P has normal vector. $\vec{T}, \vec{N}$

$\vec{B}(\frac{\pi}{2}) = \frac{1}{\sqrt{2}} \langle 1, 0, 1 \rangle \sim \underline{\langle 1, 0, 1 \rangle}$ (parallel). $\quad \vec{B}$ ↑

So an equation is.

$1(x-0) + 0(y-1) + 1(z - \frac{\pi}{2}) = 0 \Rightarrow z = -x + \frac{\pi}{2}$ □

normal vector

a point on the plane.

$(\vec{v}, \vec{a}) \longleftrightarrow (\vec{T}, \vec{N}, \vec{B})$

physics mathematics

4.

2. Motion in space; velocity and acceleration.

time position

$t \longrightarrow (x(t), y(t), z(t))$

vectors functions can also be used to describe motions.

math physics

The parameter t ⟶ time.

$\vec{r}(t)$ ⟶ position vector at time t of a particle moving through space.

curve defined by $\vec{r}(t)$ ⟶ the trajectory of the particle

$r'(t) = \vec{v}(t)$ velocity → vector.

$v = |\vec{r}'(t)| = |\vec{v}(t)|$ speed. → scalar

$\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t)$ acceleration

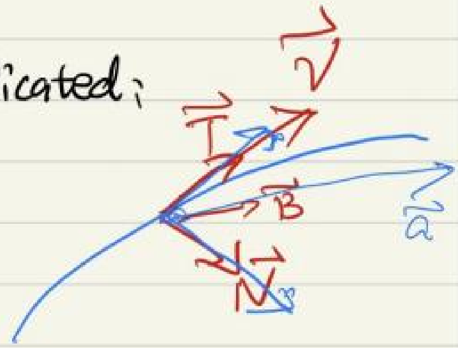
Caution	$\vec{a}(t)$	$\kappa = \left\lvert \frac{d\vec{T}}{ds} \right\rvert$
	vector	scalar.
	any parameter.	arc length.
	derivative of $\vec{r}'$	derivative of $\vec{T} = \frac{\vec{r}'}{\lvert\vec{r}'\rvert}$ unit
	☆ depends on $\vec{r}(t)$. ☆	independent of parametrization

$(\kappa \neq 0)$

At any point P on C, we have canonically defined vectors $\vec{T}$, $\vec{N}$, $\vec{B}$, we want to study the decompositions of $\vec{v}$, and $\vec{a}$ with respect to this frame.

- $\vec{v}(t) = \vec{r}'(t) = |r'(t)| \cdot \vec{T}(t)$.

The decomposition of $\vec{a}$ is more complicated;



velocity vector   speed scalar.

$$\vec{v} = v\vec{T}$$

$$\Rightarrow \vec{a} = v' = \underline{v'\vec{T}} + \underline{v\vec{T}'}. \quad \text{(Leibniz rule)} \quad ①$$

We also know that

$$\kappa = \left|\frac{d\vec{T}}{ds}\right| = \left|\frac{d\vec{T}/dt}{ds/dt}\right| = \frac{|\vec{T}'|}{v},$$

$$\text{so } |\vec{T}'| = \kappa v \qquad \vec{T}' = \underline{\kappa v \vec{N}}. \quad ②$$

$$①② \Rightarrow \boxed{\vec{a} = v'\vec{T} + \kappa v^2 \vec{N}}$$

Write $\vec{a} = a_T\vec{T} + a_N\vec{N}$, where a_T and a_N are tangential and normal components of acceleration, then.

$$a_T = v', \qquad a_N = \kappa v^2,$$

In coordinates,

$$\vec{v} \cdot \vec{a} = v\vec{T} \cdot (v'\vec{T} + \kappa v^2 \vec{N}) = vv' \quad (\vec{T} \cdot \vec{N} = 0)$$

$$\Rightarrow \boxed{a_T = v' = \frac{\vec{v} \cdot \vec{a}}{v} = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{|r'(t)|}}$$

$\vec{r}'(t)$
$\vec{r}''(t)$

$$\boxed{a_N = \kappa v^2 = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|r'(t)|^3} \cdot |r'(t)|^2 = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|r'(t)|}}$$

simple motion ↙

Example $\vec{r}(t) = \langle t^2, t^2, t^3 \rangle$.

(1) $\vec{r}'(t) = 2t\vec{i} + 2t\vec{j} + 3t^2\vec{k}$ velocity.

$\vec{r}''(t) = 2\vec{i} + 2\vec{j} + 6t\vec{k}$ acceleration.

$|r'(t)| = \sqrt{8t^2 + 9t^4}$ speed.

$2t\cdot 2 + 2t\cdot 2 + 3t^2\cdot 6t$

(2) $$a_T = \frac{\vec{r}'(t)\cdot r''(t)}{|r'(t)|} = \frac{8t + 18t^3}{\sqrt{8t^2+9t^4}}$$

(3) $$\vec{r}'(t)\times\vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t & 2t & 3t^2 \\ 2 & 2 & 6t \end{vmatrix} = 6t^2\vec{i} - 6t^2\vec{j}.$$

$$a_N = \frac{|\vec{r}'(t) + \vec{r}''(t)|}{|\vec{r}'(t)|} = \frac{6\sqrt{2}t^2}{\sqrt{8t^2+9t^4}}.$$

3, What we have learned; $\langle a, b, c\rangle$

- vectors, addition, + scalar multiplication ⇒ vector space.

geometry: dot product. ⟷ metric structure.

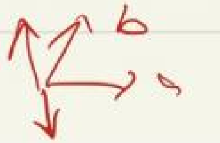
⇒: $|a|^2 = \vec{a}\cdot\vec{a}$

$\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cdot\cos\theta$. ⇐ cosine law

cross product: $\vec{a}\times\vec{b}$: $|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$,; ⇒ $\vec{a}\parallel\vec{b}$ $\vec{a}\times\vec{b} = \vec{0}$.

orthogonal to $\vec{a}, \vec{b}$; right-hand rule.

only for $\mathbb{R}^3$. ↗ compute normal vectors.



- lines and planes, simplests curves, / surfaces

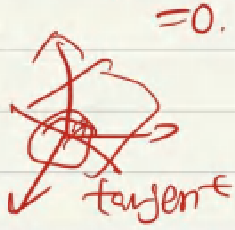
lines: a point P and direction $\vec{v}$. $\Longleftrightarrow$ parametric
(NOT unique). symmetric

planes: a point P and a normal vector. linear equation. $ax+by+cz+d=0$.

distances, angles $\Leftarrow$ $\begin{cases}\text{dot produce}\\ \text{cross product.}\end{cases}$

a,b,c → tangent point.

d → radius.



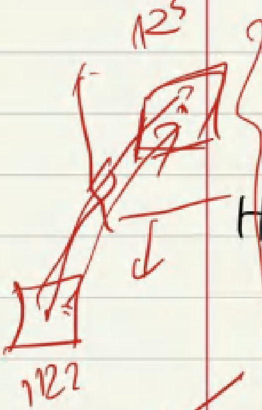
- vector functions and curves.

vector function $\longleftrightarrow$ three real functions.
$(f(t), g(t), h(t))$,
$\mathbb{R}\to\mathbb{R}$

geometric invariants: arc length, curvature. □

What we are interested in:

How to study surfaces: ← functions of several variables



curve: $\mathbb{R}\to\mathbb{R}^3$ $\longrightarrow$ surface: $\mathbb{R}^2\to\mathbb{R}^3$

+ dim2 dim2 $(s,t)\to p(s,t)$

How to study the geometry of surfaces; ← double integral

arc length $=\int|\vec{r}\,'(t)|dt$ $\longrightarrow$ area $=\iint dA$.
↑ area element

analysis

curvature $\longrightarrow$ differential geometry.
(more advanced course)

geometry

Riemannian geometry

$\mathbb{R}^m \to \mathbb{R}^n$

$f \rightsquigarrow \frac{\partial x_i}{\partial x_j}$ $\iint$

Lecture 7

1, Functions of two variables.

Definition A function f of two variables is a rule that assigns to each ordered pair of real numbers (x,y) in a set D a unique real number denoted by $f(x,y)$. The set D is the domain of f and its range is the set of values that f takes on, that is, $\{f(x,y) \mid (x,y) \in D\}$.

$\underset{\in D \subset \mathbb{R}^n}{x} \longrightarrow f(x)$

Example The temperature T at a point on the surface of the earth at any time depends on the longitude x and latitude y of the point. T is a function of two variables x and y.

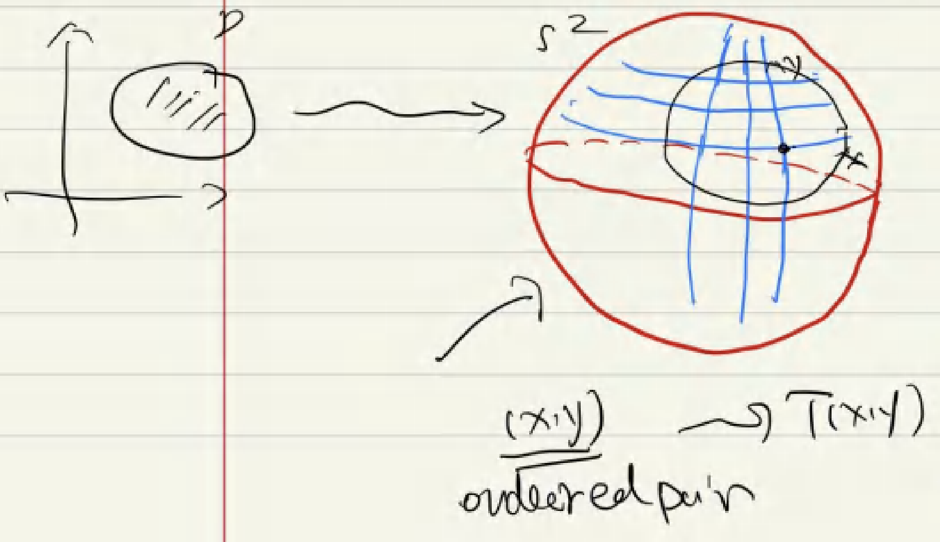
fixed ⟶

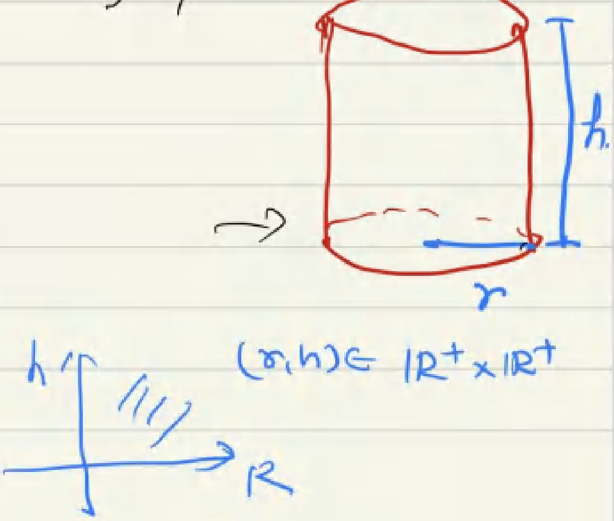
(x,y) a small domain.

Remark Note that, a sphere is not homeomorphism to an open set in $\mathbb{R}^2$; (S^2 is compact). This means that we cannot find global coordinate (x,y), such that S^2 can be identified with an open subset in $\mathbb{R}^2$. Of course S^2 is Locally an open subset in $\mathbb{R}^2$, and we can cover S^2 by several such open subsets. S^2 is the simplest example of a manifold.

⟵ geometric (object) constructed from several open sets in $\mathbb{R}^n$

Example The volume V of a circular cylinder depends on its radius r and its height h.: $V = \pi r^2 h$.





deper $\Rightarrow y=f(x)$ ↖ inde

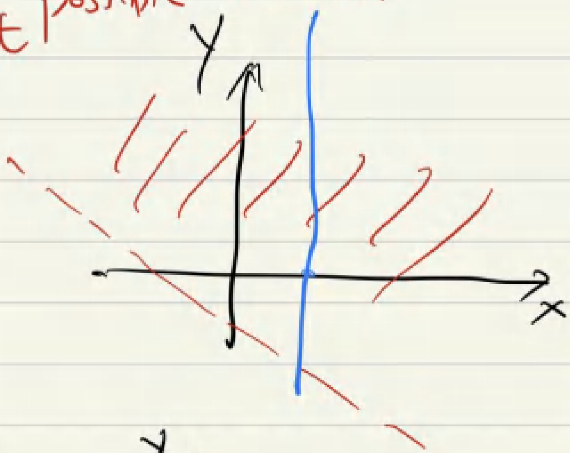
We often write $z=f(x,y)$ to make explicit the value taken on by f at the general point (x,y). The variables x and y are independent variables, and z is the dependent variable.

A function of two variables is just a function whose domain is a subset of $\mathbb{R}^2$, and whose range is a subset of $\mathbb{R}$. (x,y)

If a function f is given by a formula and no domain is specified, then the domain of f is understood to be the set of all pairs (x,y) for which the given expression is a well-defined real number. (largest possible domain)

Example $f(x,y) = \frac{\sqrt{x+y+1}}{x-1}$.

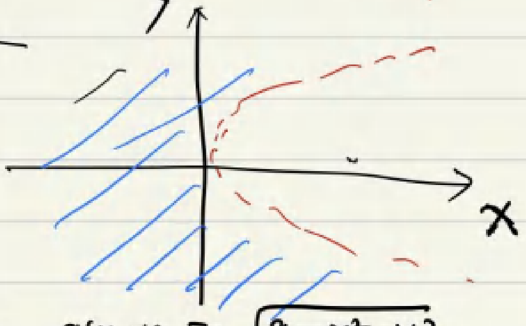
$D=\{(x,y) \mid \underline{x+y+1 \geq 0.\ x \neq 1}\}$



Example $f(x,y) = x\ln(y^2-x)$.

$D=\{(x,y) \mid y^2-x>0\}$.

$x<y^2$

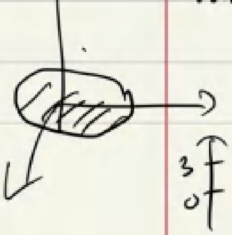


Example Find the domain and range of $g(x,y) = \sqrt{9-x^2-y^2}$.

$D=\{(x,y) \mid 9-x^2-y^2 \geq 0\} = \{(x,y) \mid x^2+y^2 \leq 9\}$, a disk,

range is $\{z \mid 0 \leq z \leq 3\} = [0,3]$.

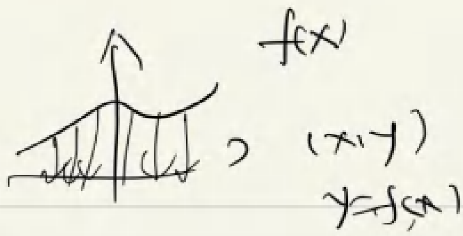
$(x,y) \in$ Disk



f(x) → range,

skip? maximat.

min — —



2, Graphs and level curves,

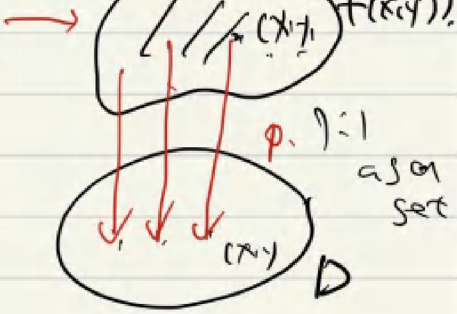
Definition. If f is a function of two variables with domain D, then the graph of f is the set of all points (x,y,z) in $\mathbb{R}^3$ such that $z=f(x,y)$ and (x,y) is in D,

The graph of a function f of one variable is a curve C with equation $y=f(x)$.

↓

The graph of a function f of two variables is a surface S with equation $z=f(x,y)$,

$z=f(x,y)$. →



f one variable.

$f(x)=k$

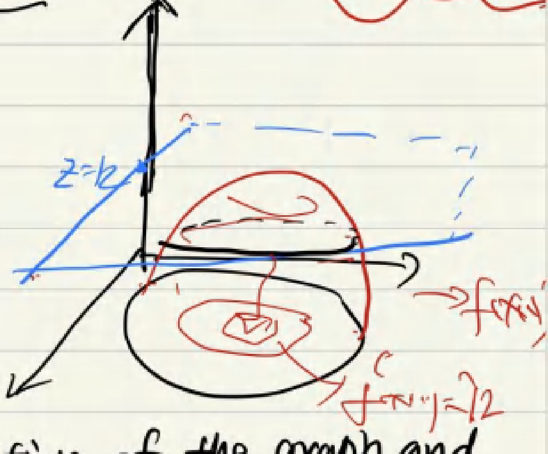
f(x) is polynomial

3 roots →

Definition The level curves of a function f of two variables are the curves with equations $f(x,y)=k$, where k is a constant (in the range of f),

if NOT in the f.

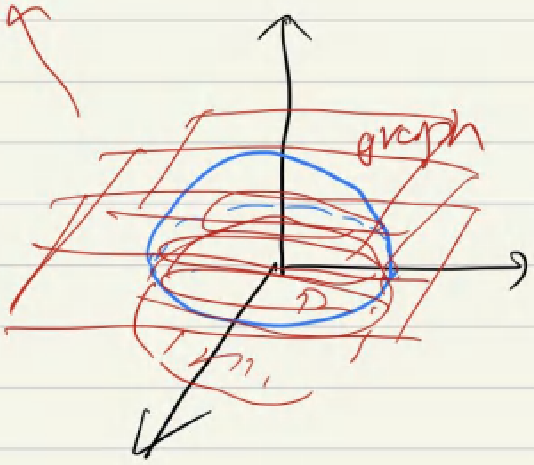
$f(x,y)=k \rightarrow \phi$



- The level curve $f(x,y)=k$, is the projection to D of the curve represented as the intersection of the graph and the plane $z=k$,

  2 surfaces ∩ → curve

- As k varies, the level curves $f(x,y)=k$ cover the domain D, Of course this level curves are pairwise disjoint, spherical coordinates



Example $g(x,y)=\sqrt{9-x^2-y^2}=z$

$x^2+y^2+z^2=9,$

$D=\{(x,y) \mid x^2+y^2\leq 9\}.$

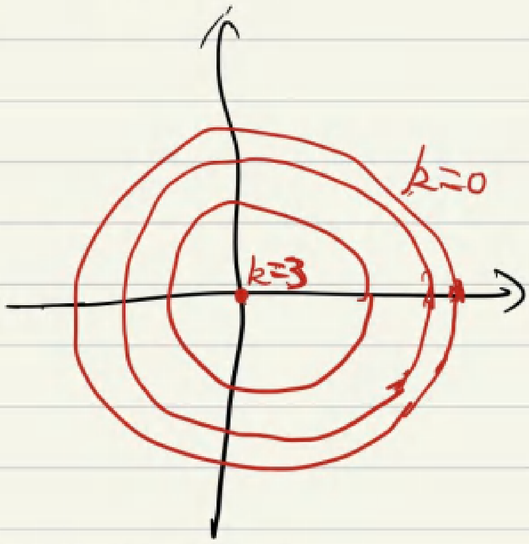
range $=[0,3].$

The graph is just the top half of the sphere

$$\{(x,y,z) \mid x^2+y^2+z^2=9\}.$$

$h(x,y)=-\sqrt{9-x^2-y^2}$ lower half.

Example Sketch the level curves of the function $g(x,y)=\sqrt{9-x^2-y^2}$. for $k=0,1,2,3.$



Solution $g(x,y)=k \Longleftrightarrow x^2+y^2=9-k^2.$

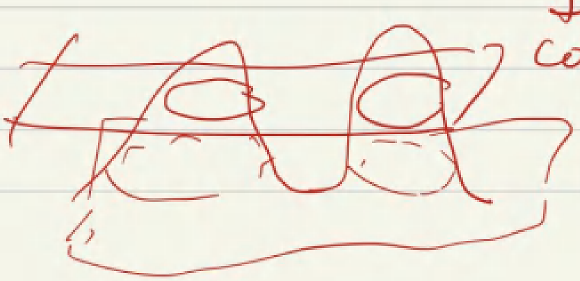
$k=0,1,2 \longrightarrow$ circles,

$k=3 \longrightarrow$ a point $(0,0)$.

Remark: level curves may NOT be a curve (dim 1), degenerate case. ⟿ level sets

level curves need not to be connected, 2 components.

level set. ← 2 components



2 mountains

definition simple.

computation difficult ← geometry of $\mathbb{R}^2$

3, Limits.

Definition Let f be a function of two variables, whose domain D includes points arbitrarily close to (a,b). Then we say that the limit of f(x,y) as (x,y) approaches (a,b) is L and we write

point in $\mathbb{R}^2$.

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

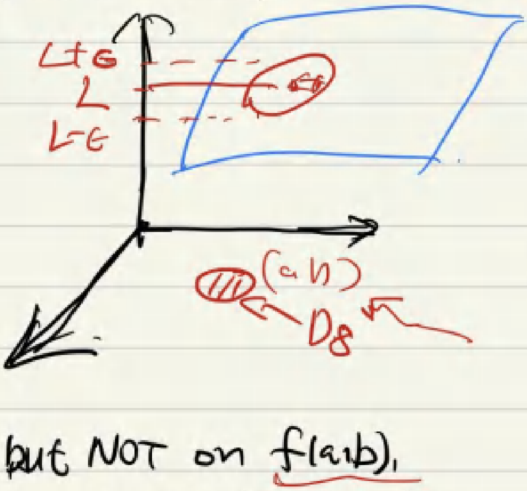
if for every number $\varepsilon > 0$, there is a corresponding number $\delta > 0$, such that

if $(x,y)\in D$, and $0 < \sqrt{(x-a)^2+(y-b)^2} < \delta$, then $|f(x,y)-L| < \varepsilon$,

Remark. The definition is quite similar to that of functions of one variable, except that replace the open intervals (a,b) of x_0, by open balls,

$0 < |x-x_0| < \delta$

The definition says that the distance between the f(x,y) and L can be made arbitrarily small by making the distance between the point (x,y) and the point (a,b) sufficiently small (but not 0).

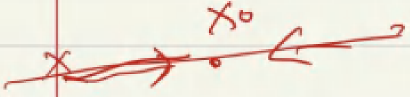


Remark The limit at the point (a,b), is determined by f(x,y) where (x,y) is sufficiently close to (a,b), but NOT on f(a,b).
We may set f(a,b) as an arbitrary number (or make f(a,b) undefined) without changing $\lim_{(x,y)\to(a,b)} f(x,y)$.

$f(x)=f_1 \cdot f_2$

on $D \setminus (a,b)$

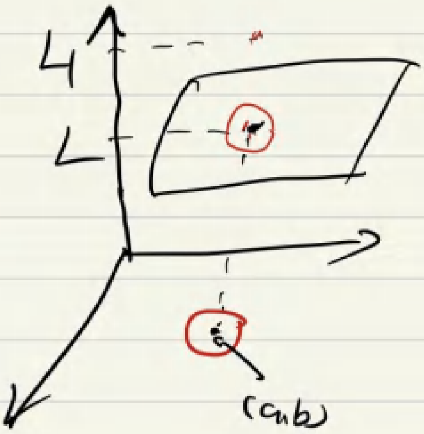
2 directions



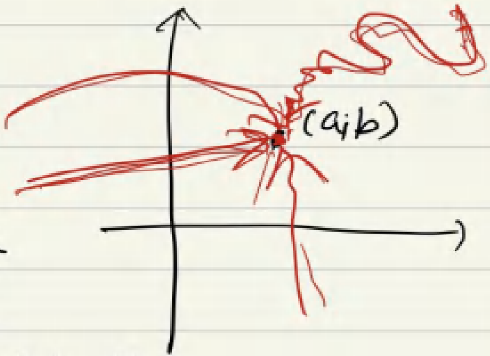
$f(a,b)=L$

$f_1(a,b)=L_1$

$f_2(a,b)$ undefined.



The geometry of $\mathbb{R}^2$ is more complicated than $\mathbb{R}^1$; we can let (x,y) approach (a,b) from an infinite number of directions in any manner whatsoever, as long as (x,y) stays within the domain of f.

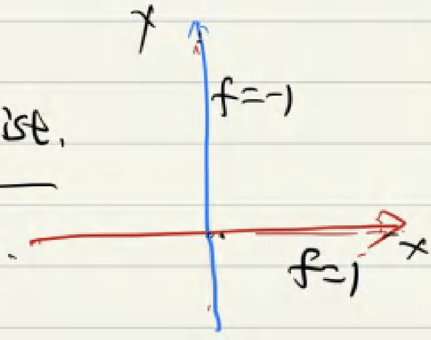


The definition of limit refers only to the distance between (x,y), and (a,b), It does not refer to the direction of approach, Thus, if we can find two different paths of approach along which the function $f(x,y)$ has different limits, then it follows that

$$\lim_{(x,y)\to(a,b)} f(x,y).$$

function of one varible.

does not exist.



Example Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

Soultelon Let $f(x,y)=\frac{x^2-y^2}{x^2+y^2}$.

approach (0,0) along the x-axis

$$f(x,0)=x^2/x^2=1 \to 1,$$

approach (0,0) along the y-axis,

$$f(0,y)=-y^2/y^2=-1 \to -1$$

□

Example If $f(x,y)=\dfrac{xy}{x^2+y^2}$, does $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist?

Solution If $y=0$, then $f(x,0)=\boxed{0,}$

$f(x,y)\to 0$ as $(x,y)\to(0,0)$ along the x-axis.

By symmetry,

$f(x,y)\to 0$ as $(x,y)\to(0,0)$ along the y-axis.

Let's now approach $(0,0)$ along another line, say $y=x$.

$$f(x,x)=\frac{x^2}{x^2+x^2}=\frac{1}{2}.$$

Therefore, $f(x,y)\to\boxed{\frac{1}{2}}$ as $(x,y)\to(0,0)$ along $y=x$,

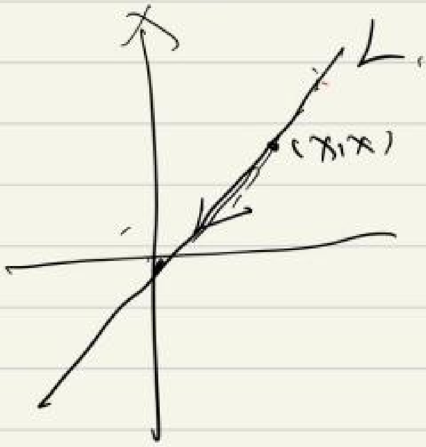
The given limit does not exise.

Remark. $$f(x,y)=\frac{y/x}{1+(y/x)^2}=\frac{x/y}{1+(x/y)^2},$$

so f is constant along every line through the origin.

Suggests –

study $f|_L$



Example If $f(x,y) = \frac{xy^2}{x^2+y^4}$, does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

Solution: Let's try to let $(x,y) \to (0,0)$ along any line through the origin.

slope

- If the line is not the y-axis, $y = mx$,

m fixed

$$f(x,y) = f(x,mx) = \frac{x(mx)^2}{x^2+(mx)^4} = \frac{m^2x}{1+m^4} \to 0$$

So $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along $y=mx$,

- If the line is the y-axis, then $x=0 \Rightarrow f(x,y)=0$.

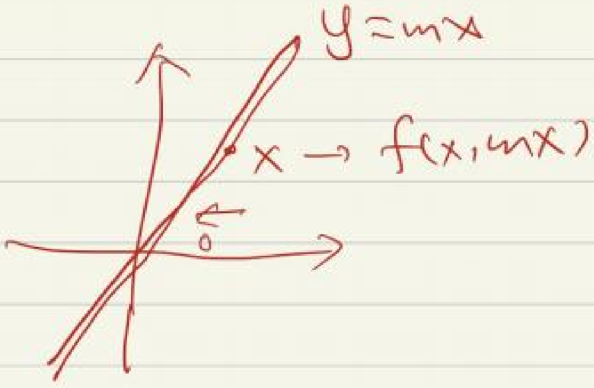
Thus f has the same limiting value along every line through the origin.

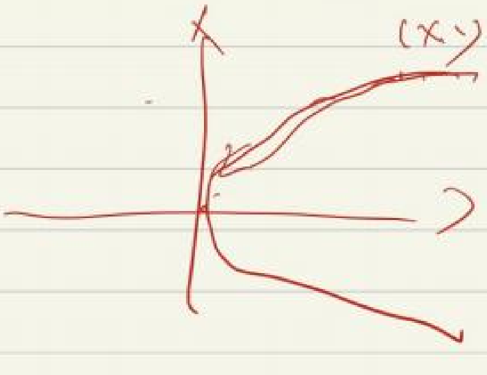
However, if we let $(x,y) \to (0,0)$ along the parabola $x=y^2$, we have,

$$f(x,y) = f(y^2,y) = \frac{y^2 y^2}{(y^2)^2+y^4} = \frac{1}{2}.$$

$f(x,y) \to \frac{1}{2}$ as $(x,y) \to (0,0)$ along $x=y^2$.

The given limit does not exist!





Limit laws can be extended to functions of two variables

$$\lim(f+g) = \lim f + \lim g, \leftarrow$$

one vari

← two

....

f-g,

f·g.

Example Prove: $\lim\limits_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0.$

Solution

Let $\varepsilon > 0$, we want to find $\delta > 0$, such that

def.

if $0 < \sqrt{x^2+y^2} < \delta$, then $\left|\frac{3x^2y}{x^2+y^2} - 0\right| < \varepsilon.$

$|f(x,y) - L| \sim \sqrt{x^2+y^2}$

that is,

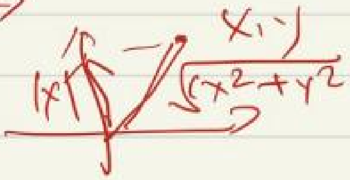
if $0 < \sqrt{x^2+y^2} < \delta$, then $\frac{3x^2|y|}{x^2+y^2} < \varepsilon,$

$x^2 \le x^2+y^2$

But $x^2/x^2+y^2 \le 1 \Rightarrow,$

$\Rightarrow$

$$\frac{3x^2|y|}{x^2+y^2} \le 3|y| \le 3\sqrt{x^2+y^2}.$$



Thus if we choose $\delta = \varepsilon/3$, and let $0 < \sqrt{x^2+y^2} < \delta$,

then $\left|\frac{3x^2y}{x^2+y^2} - 0\right| \le 3\sqrt{x^2+y^2} < 3\delta = \varepsilon.$ □

Remark, usually, it is not easy to compre $|f(x,y) - L|$ with $\sqrt{x^2+y^2}$.

or prove the existence

we calculate $\lim f(x,y)$ by writting $f(x,y)$ as sum, product, composition of "simple functions".

Example $$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$$

Solution when $(x,y) \neq (0,0)$.

$$x^2+y^2 = x^2+y^2+1-1$$
$$= \left(\sqrt{x^2+y^2+1}+1\right)\left(\sqrt{x^2+y^2+1}-1\right).$$

$$\Rightarrow \quad \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} = \boxed{\sqrt{x^2+y^2+1}+1}.$$

Its limit at $(0,0) = \sqrt{1}+1 = 2$.

□

$$\left(\frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} - 1 \right| \leq C\sqrt{x^2+y^2}$$

limit

functions of 2 variable

limit $\xrightarrow{\text{DNE}}$ restriction to curves. 1,

$\xrightarrow[\text{axis}]{}$ simple

Lecture 8.

1, Continuity,

Definition A function f of two variables is called continuous at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

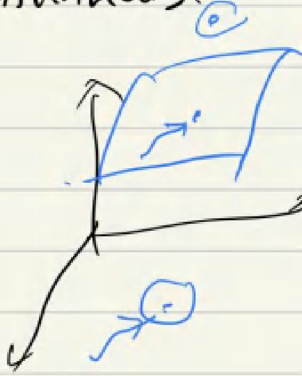
We say f is continuous on D if f is continuous at every point (a,b) in D.

if the point (x,y) changes by a small amount, then the value of f(x,y) changes by a small amount. ~ chain

A surface that is the graph of a continuous function has no pole or break.

- sums, differences, products, quotients of continuous functions are continuous on their domains,
- Composition: $f: \mathbb{R}^2 \to \mathbb{R}$, $g: \mathbb{R}\to\mathbb{R}$; both continuous.

  ☆ $\Rightarrow$ $g\circ f: \mathbb{R}^2\to\mathbb{R}$ continuous.



Example : polynomial function of two variables:

$$\sum_{m,n} c_{m,n} x^m y^n. \quad (m,n \geq 0)$$

finite sum

$$f(x,y) = x^4 + 5x^3y^2 + 6xy^4 - 7y + 6.$$

rational function: ratio of polynomials.

$f: \mathbb{R}^2 \to \mathbb{R}$

$$g(x,y) = \frac{2xy+1}{x^2+y^2}.$$

$g. \mathbb{R} \to \mathbb{R}.$

algebraic

↘

polynomial ←→ series ← limit / analysis

2,

If a function f is not defined at a point (a,b), but $\lim_{(x,y)\to(a,b)} f(x,y) = L$ exists, then we may extend f to a function continuous at (a,b) by setting $f(a,b)=L$.

Example Let

$$f(x,y) = \begin{cases} \dfrac{3x^2y}{x^2+y^2} & (x,y) \neq (0,0) \\ \underline{\underline{0}} & (x,y) = (0,0) \end{cases}$$

$\leq C\sqrt{x^2+y^2}$

We know f is continuous, for $(x,y) \neq 0$. $\leftarrow$ rational function

Also $\lim_{(x,y)\to(0,0)} f(x,y) \overset{?}{=} 0$.

$\Rightarrow$ f is continuous on $\mathbb{R}^2$.

$\mathbb{R}^2 \setminus 0$.

"extension"

Remark. $x^2+y^2=0 \iff (x,y)=(0,0)$.

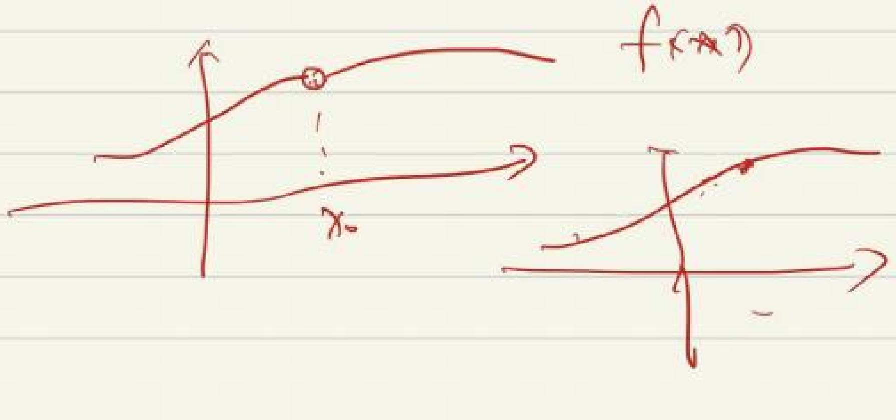
$f(x,y)$ has only one "bad point",

for general $f = \dfrac{g(x,y)}{h(x,y)}$,

pole

f is NOT defined at $\{(x,y) \mid h(x,y)=0\}$, a curve.

Things become more complicated,



$(x,y,z) \longrightarrow f(x,y,z)$

$(x,y,z) \to (a,b,c)$ 3,

$\uparrow$

$\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$

2, Functions of three variables,

The definitions, limits, continuity is the same as, functions of two variables,

Remark : Can not draw graphs for functions of three variables

temperatura function : $T(x,y,z)$.

density $\rho(x,y,z)$ $\boxed{w = f(x,y,z)}$

$\cap$

$\mathbb{R}^4$

Remark. Level surfaces: $f(x,y,z)=k$.

Example $f(x,y,z) = x^2+y^2+z^2$,

level surfaces:

$$f(x,y,z) = x^2+y^2+z^2 = k, \Longleftrightarrow \text{spheres}.$$

□

---

①. Functions of several variables

$\Updownarrow$

one variable

derivate ⟺ integral

$\uparrow$

scales

restriction to curves.

geometry

② Local properties of functions

$\Updownarrow$

Linear algebra

{ tangent space. tangent map.

Linear algebra.

$$f(x) \to f'(a)$$

$$\to \underbrace{f'(x)}_{\text{function}}$$

3, Partial derivatives,

If f is a function of two variables x and y, suppose we let only x vary while keeping y fixed. say y=b, where b is a constant. Then we are really considering a function of a single variable x, namely, $g(x) = f(x,b)$. If g has a derivative at a, then we call it the partial derivative of f with respect to x at (a,b). and denote it by $f_x(a,b)$.

$$\boxed{f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}}$$

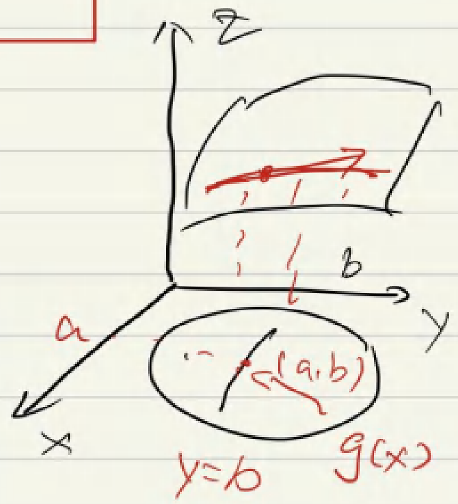
The partial derivative of f with respect to y at (a,b) is defined similarly.

$\boxed{x=a}$

$$\boxed{f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}}$$

$h(y) = f(a,y)$

Let the point (a,b) vary, f_x and f_y becomes functions of two variables:



$$\boxed{\begin{aligned} f_x(x,y) &= \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \\ f_y(x,y) &= \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h} \end{aligned}}$$

$z = f(x,y)$

Notations $f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$ ...

first variable

To find f_x, regard y as constant and differentiate $f(x,y)$ with respect to x.

$x^3 \longrightarrow 3x^2$.

$x^2 y^3 \longrightarrow y^3 \cdot 2x$

$2y^2 \longrightarrow 0$

Example $f(x,y) = x^3 + x^2y^3 - 2y^2$

$$\underline{f_x(x,y)} = 3x^2 + \underline{2xy^3} \qquad f_x(2,1) = 16.$$

$$\underline{f_y(x,y)} = 3x^2y^2 - 4y \longrightarrow \qquad f_y(2,1) = 8.$$

$x^3 \longrightarrow 0.$

$x^2y^3 \longrightarrow x^2 \cdot 3y^2$

$-2y^2 \longrightarrow -4y.$

Example $f(x,y) = \sin\left(\frac{x}{1+y}\right)$.

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right)\frac{\partial}{\partial x}\left(\frac{x}{1+y}\right) = \cos\left(\frac{x}{1+y}\right) \cdot \frac{1}{1+y}.$$

$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right)\frac{\partial}{\partial y}\left(\frac{x}{1+y}\right) = -\cos\left(\frac{x}{1+y}\right)\frac{x}{(1+y)^2}.$$

$\frac{x}{c} = \dot{c}$

$$\frac{\partial}{\partial y}\left(\frac{1}{1+y}\right) = -\frac{y' \quad 1}{(1+y)^2}$$

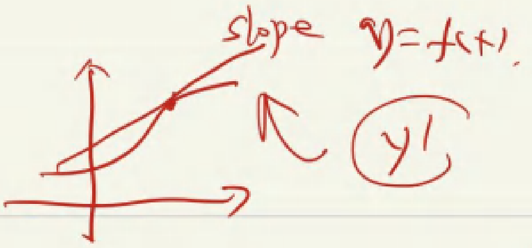
Properties:

$$\frac{\partial}{\partial x}(f+g) = \frac{\partial}{\partial x}f + \frac{\partial}{\partial x}g.$$

$$\frac{\partial}{\partial x}(fg) = \frac{\partial}{\partial x}f \cdot g + f \cdot \frac{\partial}{\partial x}g.$$

chain rule? more complicated . . .

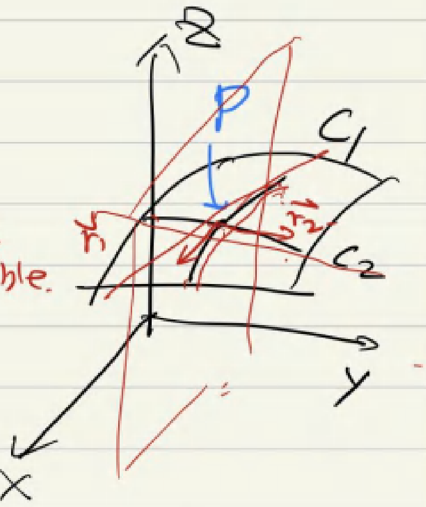
partial <——> total derivative



4 Tangent planes

graph

Suppose a surface S has equation. $z=f(x,y)$, where f has continuous first partial derivatives, and let $P(x_0,y_0,z_0)$. be a point on S.



The intersection of S with the plane $y=y_0$. is a curve C_1: its equation is,

$(x, y_0, f(x,y_0)),)$. ← graph of $f(x,y_0)$ ← one varible.

and its tangent vector $\vec{r_1}$ at P_0 is.

$\frac{dx}{dx}$ $(1, 0, \frac{\partial f}{\partial x}(x_0,y_0))$.

Similarly, we get another vector $\vec{r_2} = (0,1, \frac{\partial f}{\partial y}(x_0,y_0))$.

Denote by T_1 (resp. T_2) the tangent line of C_1 (resp. C_2) at P.

The tangent plane to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2.

What is the equation of the tangent plane?

- A point $P= (x_0,y_0,z_0)$.

$$y = f(x)$$

$$\Downarrow$$

tangent line at (x_0, y_0)

is $\boxed{y - y_0 = f'(x_0)(x - x_0)}$

- normal vector.

$$\vec{n} = \vec{r_1} \times \vec{r_2}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x}\vec{i} - \frac{\partial f}{\partial y}\vec{j} + \vec{k}.$$

So the equation is. constant

$$-\frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) - \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) + (z - z_0) = 0.$$

$$\Longleftrightarrow \boxed{z - z_0 = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0).}$$

☆

Example Find the tangent plane to the elliptic paraboloid.

$$z = 2x^2 + y^2.$$

at the point $(1, 1, 3)$.

Solution. S is the graph of $f = 2x^2 + y^2$. $P = (1, 1, f(1,1))$

$$\frac{\partial f}{\partial x}(x, y) = 4x \qquad \frac{\partial f}{\partial x}(1,1) = 4.$$

$$\frac{\partial f}{\partial y}(x, y) = 2y, \qquad \frac{\partial f}{\partial y}(1,1) = 2,$$

So the equation:

$$z - 3 = 4(x-1) + 2(x-1).$$

$$\Longleftrightarrow \qquad z = 4x + 2y - 3.$$

$y=f(x) \longrightarrow \underline{\underline{f'}} \longrightarrow f'' \longrightarrow \cdots$

$f^{(n)}(x)$

4. Higher derivatives.

$$\begin{tikzcd}
 & & (f_x)_x \\
 & f_x \arrow[ur] \arrow[r] & (f_x)_y \\
f \arrow[ur] \arrow[dr] & & \\
 & f_y \arrow[r] \arrow[dr] & (f_y)_x \\
 & & (f_y)_y
\end{tikzcd}$$

$\cdots 2^n \cdots$

8. third

partial derivatives

$\underline{\text{second partial derivatives}}$

notation $(f_x)_x = \underline{f_{xx}} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \underline{\frac{\partial^2 f}{\partial x^2}} = \frac{\partial^2 z}{\partial x^2}$

$(f_x)_y = \underline{f_{xy}} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \underline{\frac{\partial^2 f}{\partial y \partial x}} = \frac{\partial^2 z}{\partial y \partial x}$

Example $f(x,y) = x^3 + x^2y^3 - 2y^2$

$f_x(x,y) = \underline{3x^2 + 2xy^3}$ $\quad f_y(x,y) = 3x^2y^2 - 4y$

$\underline{f_{xx}} = \frac{\partial}{\partial x}(3x^2+2xy^3) = 6x + 2y^3$

$3x^2 \longrightarrow 6x.$

$2xy^3 \longrightarrow 2y^3.$

$$\left.\begin{aligned} f_{xy} &= \frac{\partial}{\partial y}(3x^2+2xy^3) = 6xy^2 \\ f_{yx} &= \frac{\partial}{\partial x}(3x^2y^2-4y) = 6xy^2 \end{aligned}\right\}$$

$f_{yy} = \frac{\partial}{\partial y}(3x^2y^2-4y) = 6x^2y - 4.$

"good" functions

Mixed partial theorem

Suppose f is defined on a disk D that contains the point (a,b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a,b) = f_{yx}(a,b).$$

$f: 2, 3, \ldots, m$

5. General functions $F: \mathbb{R}^m \to \mathbb{R}^n$.

We may defines with. m-variables: $\mathbb{R}^m \to \mathbb{R}^1$.

$$f = f\underline{(x_1 \cdots x_m)}.$$

n such funcitons $f_1 \cdots f_n$ defines a map.

$$F: \mathbb{R}^m \to \mathbb{R}^n.$$

f_n -- components --

$$F: (x_1 \cdots x_m) = (f_1(x_1 \cdots x_m), \cdots f_n(x_1 \cdots x_m)).$$

- limits $\mathbb{R}^m \to \mathbb{R}$

f: $d((x_1 \cdots x_m), (y_1 \cdots y_m)) = \sqrt{\sum_i (x_i - y_i)^2}$.

$\vec{F}$: limit of components.

$\mathbb{R} \to \mathbb{R}^m$

$\vec{r}$

$\uparrow$

real

Continuity:

- partial derivatives:

$$\frac{\partial f}{\partial x_i}(x_1 \cdots x_m) = \lim_{h \to 0} \frac{f(x_1 \cdots x_i + h, \cdots x_m) - f(x_1 \cdots x_m)}{h}.$$

$\uparrow$

derivative on x_i -direction

6. Matrices defined by partial derivatives.

- The Gradient vector

$$f : \mathbb{R}^m \to \mathbb{R}.$$

$$\nabla f := \left(\frac{\partial f}{\partial x_1}, \cdots \frac{\partial f}{\partial x_m} \right)$$

Example $f(x,y,z) = e^{xy} \ln z.$

$$\frac{\partial f}{\partial x} = y e^{xy} \ln z. \quad \frac{\partial f}{\partial y} = x e^{xy} \ln z. \quad \frac{\partial f}{\partial z} = \frac{e^{xy}}{z},$$

3 functions

$$\nabla f = \left(y e^{xy} \ln z,\ x e^{xy} \ln z,\ \frac{e^{xy}}{z} \right).$$

vector.

- The Hessian matrix.

$$f : \mathbb{R}^m \to \mathbb{R},$$

$$H(f) = \begin{bmatrix} f_{1,1} & \cdots & f_{1,m} \\ \vdots & \ddots & \\ f_{m,1} & \cdots & f_{m,m} \end{bmatrix}.$$

second partial derivatives

Remark By mixed partials theorem, H(f) is symmetric:

$$f_{i,j} = f_{j,i}. \quad \text{or} \quad H(f)^T = H(f).$$

along diagonal

Example. $f(x,y) = x^3 + x^2y^3 - 2y^2,$

$$H(f) = \begin{pmatrix} 6x + 2y^3 & 6xy^2 \\ 6xy^2 & 6x^2y - 4 \end{pmatrix}$$

- The Jacobian Matrix.

$F: \mathbb{R}^m \longrightarrow \mathbb{R}^n$.

$\vec{F} = (f_1(x_1 \cdots x_m) \cdots f_n(x_1 \cdots x_m))$.

$$J(F) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_m} \end{pmatrix} \begin{matrix} \longleftarrow \nabla f_1 \\ \vdots \\ \longleftarrow \nabla f_n \end{matrix}$$

gradient vector

Example. $F(x, y) = (x^2 y, x - y^2)$.

Solution Let $f_1 = x^2 y$. $f_2 = x - y^2$.

$\frac{\partial f_1}{\partial x} = 2xy \qquad \frac{\partial f_1}{\partial y} = x^2$,

$\frac{\partial f_2}{\partial x} = 1 \qquad \frac{\partial f_2}{\partial y} = -2y$

$$\Rightarrow \quad J(F) = \begin{pmatrix} 2xy & x^2 \\ \hline 1 & -2y \end{pmatrix}$$

□

partitial derivaties.

$\frac{\partial}{\partial x}(f+g) = \frac{\partial}{\partial x}f + \frac{\partial}{\partial x}g$ 1、

Lecture 8. the Chain rule

chain

1, Some linear algebra

(functions.)

A matrix is a rectangular array of numbers, arranged in rows and columns.

A matrix with m rows and n columns is called a m by n matrix. The set of such matrices: $M_{m\times n}(\mathbb{R})$.

Example A two by three matrix:

$$\begin{bmatrix} 1 & 9 & -13 \\ 20 & 5 & -6 \end{bmatrix}.$$

2 rows. 3 colums.

6

- Matrices of the same size form a vector space.

m×n

addition and scalar multiplication is defined naturally:

Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10. & 12 \end{bmatrix} \Leftarrow$$

add components

$$2\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

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- multiplication : $M_{m\times n}(\mathbb{R}) \times M_{n\times r}(\mathbb{R}) \longrightarrow M_{m\times r}(\mathbb{R})$.

$$m\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}\begin{bmatrix} b_{11} & \cdots & b_{1r} \\ \vdots & & \\ b_{n1} & \cdots & b_{nr} \end{bmatrix} n = \begin{bmatrix} \sum_{i=1}^{n} a_{1i}\cdot b_{i1} & \cdots & \sum_{i=1}^{n} a_{1i}\cdot b_{ir} \\ \vdots & & \\ \sum_{i=1}^{n} a_{mi} b_{i1} & \cdots & \sum_{i=1}^{n} a_{mi} b_{ir} \end{bmatrix} m$$

n

n

m r

r

a_{ij} — i-th j-th

m×n

c. col. colm

of c ← i-th row · j-th of B

We may regard a matrix $A \in M_{m\times n}(\mathbb{R})$ as m row, vectors $A = \begin{pmatrix} \vec{a_1} \\ \vdots \\ \vec{a_m} \end{pmatrix}$, and a matrix $B \in M_{m\times n}(\mathbb{R})$ as r column vectors $B = (\vec{b_1} \cdots \vec{b_r})$, Let $C = AB$.

Then the (i,j)-th entry of C:

$$\boxed{c_{ij} = \vec{a_i} \cdot \vec{b_j}}$$

$A \in M_{2\times 2}(\mathbb{R})$

$B \in M_{2\times 3}(\mathbb{R})$

Example $A = \begin{bmatrix} 2 & 4 \\ -3 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 6 & -2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$.

$$C = A \cdot B = \begin{bmatrix} 12 & 0 & 14 \\ -18 & \textcircled{6} & -3 \end{bmatrix}$$

$(2,4)\cdot(6,0)$

$(2,4)\cdot(-2,1)=0$

($m\times r$. dot products)

Remark $A \cdot B$ is defined only when the number of columns of A = the number of rows of B.

In particular, $A \cdot B$ is always defined if $A, B \in M_{n\times n}(\mathbb{R})$. $M_{n\times n}(\mathbb{R})$ is a ring. $\to (+, -, \times)$ =

$\boxed{[(\ )]}$

Remark Matrix multiplication is associative:

$$(A \cdot B) \cdot C = A \cdot (B \cdot C).$$

But it is NOT commutative.

- $M_{2\times 3}(\mathbb{R}) \times M_{3\times 2}(\mathbb{R}) \longrightarrow M_{2\times 2}(\mathbb{R})$,

while

A B

$$M_{3\times 2}(\mathbb{R}) \times M_{2\times 3}(\mathbb{R}) \longrightarrow M_{3\times 3}(\mathbb{R})$$

B , A

Even two matrices A, B are of the $\underbrace{\text{same size}}$, A·B is NOT necessarilly equal to B·A:

$$\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}0&0\\1&0\end{bmatrix}=\begin{bmatrix}\textcircled{1}&0\\0&0\end{bmatrix}$$

$$A\cdot B \neq B\cdot A$$

$$\begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}=\begin{bmatrix}0&0\\0&\textcircled{1}\end{bmatrix}.$$

□

Remark. Cancellation law does NOT hold:

$$A\cdot B = A\cdot C \not\Rightarrow B=C.$$

$$\begin{bmatrix}0&1\\0&0\end{bmatrix}\cdot\underbrace{\begin{bmatrix}0&1\\0&0\end{bmatrix}}=0=\begin{bmatrix}0&1\\0&0\end{bmatrix}\underbrace{\begin{bmatrix}0&0\\0&0\end{bmatrix}},$$

$$0 \rightarrow \begin{bmatrix}0&0\\0&0\end{bmatrix}$$

But $\begin{bmatrix}0&1\\0&0\end{bmatrix}\neq\begin{bmatrix}0&0\\0&0\end{bmatrix}$.

$$\begin{pmatrix}x\\y\\z\end{pmatrix}$$

2, Why matrix multiplication?

An m×n matrix A defines a linear map: $\mathbb{R}^n \to \mathbb{R}^m$:

a point in $\mathbb{R}^n$ can be represented as a $\underbrace{\text{column vector}}$, $\underbrace{\vec{b}}$, then we can define

$$M_{m\times n}(\mathbb{R}) \times M_{n\times 1}(\mathbb{R}) \longrightarrow M_{m\times 1}(\mathbb{R}),$$
$$(A\ ,\ \vec{b}) \longrightarrow A\vec{b}. \rightarrow (\ )\vdots m$$

So $A\vec{b}$ is a column vector of length m, thus represents a point in $\mathbb{R}^m$.

$$\underset{\mathbb{R}^n}{(\ )\vdots n} \longrightarrow \underset{\searrow\ \mathbb{R}^m}{(\ )^m}$$

Example

$$\overset{A}{\begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}}\begin{bmatrix}x\\y\end{bmatrix}^{\in\mathbb{R}^2} = \begin{bmatrix}x+2y\\3x+4y\\5x+6x\end{bmatrix}^{\in\mathbb{R}^3}$$

In particular,

$$\begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\3\\5\end{bmatrix};\quad \begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}2\\4\\6\end{bmatrix}$$

$e_1 \quad e_2$

So the image of the two canonical vectors are just the column vectors of A. □

Now suppose we have two matrices $A\in M_{m\times n}(\mathbb{R})$, $B\in M_{n\times r}(\mathbb{R})$. B defines a linear map: $\mathbb{R}^r\longrightarrow\mathbb{R}^n$, A defines a lineamap $\mathbb{R}^n\longrightarrow\mathbb{R}^m$, Use A(resp. B) to represent the corresponding linear maps.

Then their composition $A\circ B$ is a linear map: $\mathbb{R}^r\longrightarrow\mathbb{R}^m$, so can be represented by a matrix $C\in M_{m\times r}(\mathbb{R})$.

We show that $C = A\cdot B$!

linear map uniquely determined by $f(e_i)$

we consider, $A\circ B(e_i)$, where $e_i = \begin{pmatrix}0\\\vdots\\1\\\vdots\\0\end{pmatrix}\leftarrow i \in \mathbb{R}^r$.

Then $$B(e_i) = \begin{pmatrix}b_{1i}\\\vdots\\b_{ni}\end{pmatrix} = \sum_k b_{ki} f_k. \leftarrow \begin{pmatrix}0\\\vdots\\1\\\vdots\\0\end{pmatrix}\leftarrow k \in \mathbb{R}^n$$

$\Leftarrow$ (i-th column vector), of B

But the $$A\cdot B(e_i) = \sum_k b_{ki}\begin{pmatrix}a_{1k}\\\vdots\\a_{nk}\end{pmatrix}$$

$\leftarrow$ (k-th column vector)

$$= \begin{pmatrix}\sum_k a_{1k}b_{ki}\\\vdots\\\sum_k a_{nk}b_{ki}\end{pmatrix} \overset{!!!}{=} (\text{i-th column of } C) \ ★$$

$C \rightsquigarrow A\cdot B$

In conclusion,

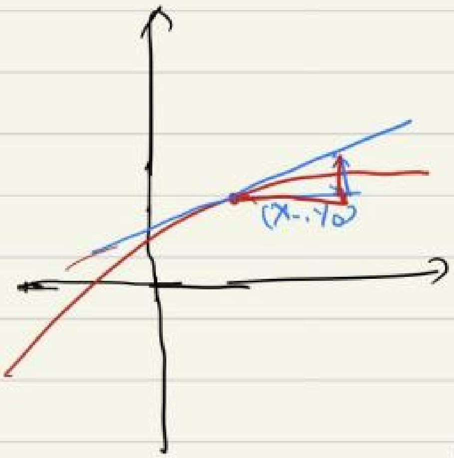
$$\text{matrix} \overset{\text{algebra}}{\longleftrightarrow} \overset{\text{analysis}}{\text{linear maps.}}$$

matrix multiplication $\longleftrightarrow$ composition of linear maps.

3 Linear approximation

① $y = f(x)$,

$f'(x)$



The tangent line at (x_0, y_0) is,

$$y - y_0 = f'(x_0)(x - x_0),$$

The tangent line is the best line approximating f at the point (x_0, y_0)

$f'(x_0)$ can be considered as a linear map:

$$(x - x_0) \rightsquigarrow y - y_0 = f'(x_0)(x - x_0),$$

$f(x+\Delta x) - f(x)$

It tells us how to get the increment of y from the increment of x in this linear approximation.

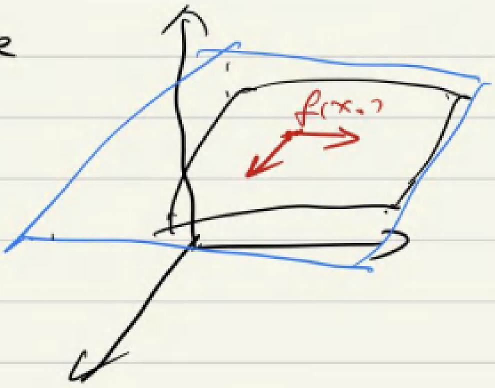
Δx

② Now $f = f(x, y)$,

The tangent plane:

$$\boxed{z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$$

The linear function whose graph is the tangent plane at (x_0, y_0), namely,



$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

is called the linearization of f at (x_0, y_0) and the approximation

$$f(x,y) \approx L(x,y).$$

is called the linear approximation or the tangent plane approximation of f at (x_0, y_0).

Suppose x changes from x_0 to $x_0+\Delta x$, and y changes from y_0 to $y_0+\Delta y$, Then the corresponding increment of z is

$$\Delta(z) = \underline{f(x_0+\Delta x, y_0+\Delta y)} - \underline{f(x_0, y_0)}.$$

Definition If $z = f(x,y)$, then f is differentiable at (x_0, y_0). if Δz can be expressed in the form.

$$\Delta z = \underline{f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y} + \boxed{\underbrace{\varepsilon_1\Delta x + \varepsilon_2\Delta y}_{\to 0}}$$

where $\varepsilon_i \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

A differentiable function is one for which the linear approximation is a good approximation when (x,y) is near (x_0, y_0)

Theorem If the partial derivatives f_x and f_y exist near (x_0, y_0) and are continuous at (x_0, y_0), then f is differentiable at (x_0, y_0).

↖ "good functions"

Example $f(x,y) = xe^{xy}$

The partial derivatives are

$$f_x(x,y) = e^{xy} + xye^{xy}$$
$$f_y(x,y) = x^2e^{xy}.$$

Both f_x and f_y are continuous functions, so f is differentiable. (at any point)

At $(1,0)$, $f_x(1,0) = 1$, $f_y(1,0) = 1$

The Linearilization is,

$$L(x,y) = \underline{1} + 1(x-1) + 1 \cdot y = x + y,$$

The corresponding linear approximation is

graph of f →

$$\underline{xe^{xy}} \simeq x + y.$$

← graph of the tangent plane.

$$f(1.1, -0.1) \approx 1.1 - 0.1 = \underline{1}.$$

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③ Now suppose $F: \underline{\mathbb{R}^n} \longrightarrow \underline{\mathbb{R}^m}$ $(\mathbb{R}^m \to \mathbb{R}^n)$

$$F = (f_1(x_1 \cdots x_n), \quad \cdots \quad f_m(x_1 \cdots x_n)).$$

Assume $f_i(x_1 \cdots x_n)$ are "good" functions, say, they are smooth, or at least C^1.
(all parital derivatives exist and are continuous)

- $f_i(x_1+\Delta x_1 \cdots\ x_n+\Delta x_n) - f_i(x_1 \cdots x_n)$

$$\approx \frac{\partial f_i}{\partial x_1}(x_1 \cdots x_n)\cdot\Delta x_1 + \cdots + \frac{\partial f_i}{\partial x_n}(x_1 \cdots x_n)\cdot\Delta x_n$$

$$= \langle \nabla f_i, \overrightarrow{\Delta x} \rangle.$$

- considering all components, we get,

$$F(x_1+\Delta x_1 \cdots\ x_n+\Delta x_n) - F(x_1 \cdots x_n).$$

$$\approx \begin{pmatrix} \nabla f_1 \\ \vdots \\ \nabla f_n \end{pmatrix} \cdot \begin{pmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{pmatrix}. \leftarrow \text{matrix multiplication}$$

$$= \underline{J(F)_{(x_1 \cdots x_n)}} \cdot \begin{pmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{pmatrix}$$

Thus, $J(F)_{(x_1 \cdots x_n)}$ tells you how to compute the linear approximation of F:

the increment $F(x_1+\Delta x_1 \cdots\ x_n+\Delta x_n) - F(x_1 \cdots x_n)$ is approximate to.

$$J(F)_{(x_1 \cdots x_n)} \cdot \begin{pmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{pmatrix}. \qquad \square$$

4 The chain rule.

Suppose we have functions $F: \mathbb{R}^n \to \mathbb{R}^m$, $G: \mathbb{R}^r \to \mathbb{R}^n$:

$$F = (f_1(y_1 \cdots y_n) \cdots \quad f_m(y_1 \cdots y_n)),$$
$$G = (g_1(x_1 \cdots x_r) \cdots \quad g_n(x_1 \cdots x_r)).$$

$$\begin{tikzcd} \mathbb{R}^r \arrow[r, "G"'] & \mathbb{R}^n \arrow[r, "F"'] & \mathbb{R}^m \end{tikzcd}$$

Their composition $F \circ G$ is a map $\mathbb{R}^r \to \mathbb{R}^m$.

Assume that all functions f_i, g_j are "good" functions, then

$$G(x_1+\Delta x_1 \cdots x_r+\Delta x_r) - G(x_1 \cdots x_r) \approx J(G)_{(x_1 \cdots x_r)} \begin{pmatrix} \Delta x_1 \\ \vdots \\ \Delta x_r \end{pmatrix}$$

$$F(y_1+\Delta y_1 \cdots y_n+\Delta y_n) - F(y_1 \cdots y_n) \approx J(F)_{(y_1 \cdots y_n)} \begin{pmatrix} \Delta y_1 \\ \vdots \\ \Delta y_n \end{pmatrix}$$

$$(\Delta y_1 \cdots \Delta y_n) = G(x_1+\Delta x_1 \cdots x_r+\Delta x_r) - G(x_1 \cdots x_r).$$

$$\Rightarrow F \circ G(x_1+\Delta x_1 \cdots x_r+\Delta x_r) - F \circ G(x_1 \cdots x_r)$$

$$\approx J(F)_{(y_1 \cdots y_n)} \cdot J(G)_{(x_1 \cdots x_r)} \begin{pmatrix} \Delta x_1 \\ \vdots \\ \Delta x_r \end{pmatrix}$$ matrixes.

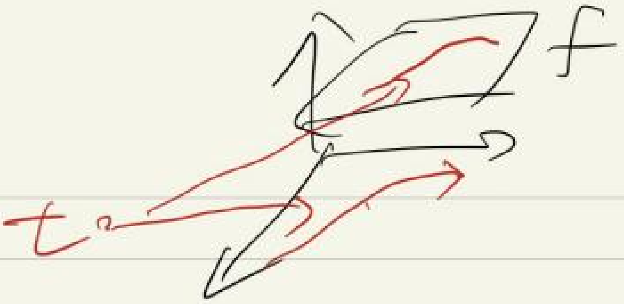
$$\Rightarrow \boxed{J(F \circ G)_{(x_1 \cdots x_r)} = J(F)_{(y_1 \cdots y_n)} \cdot J(G)_{(x_1 \cdots x_r)}}$$

□

Linear approximation. 2 worlds

bridge

- smooth functions $\mathbb{R}^n \to \mathbb{R}^m$ ⟶ Jacobian matrices; tangent maps; $\mathbb{R}^n \to \mathbb{R}^n$
- composition of smooth functions; ⟶ Product of Jacobian matrices



5. Some examples

① $z = f(x,y)$ differentiable. $\mathbb{R}^2 \to \mathbb{R}^1$

$\vec{r}(t) = (x(t), y(t))$ $\mathbb{R}^1 \to \mathbb{R}^2$

$\mathbb{R} \to \mathbb{R}$

$z(t) = f(x(t), y(t))$ is just the restriction of f on the curve represented by $\vec{r}(t)$,

The Jacobian of $\vec{r}(t)$ is.

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} \leftarrow$$

rows are ~~compon~~ gradient of c

← at t

The Jacobian of $f(x,y)$ is.

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \leftarrow$$

gradient

← at x, y,

So

$$\boxed{\begin{aligned} \frac{dz}{dt} &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)_{(x,y)} \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} \\ &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \end{aligned}}$$

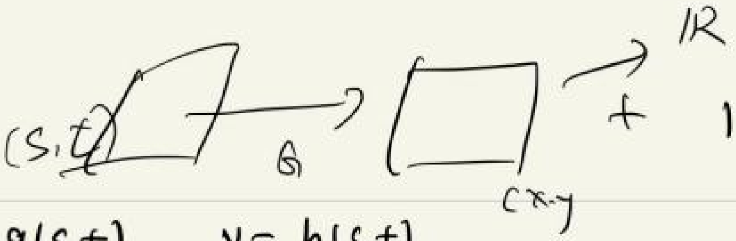
Example $z = x^2y + 3xy^4$, $x = \sin 2t$, $y = \cos t$, function t

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)$$

when $t=0$, $x = 0$, $y = 1$

$$dz/dt\big|_{t=0} = (0+3)(2\cos 0) + (0+0)(-\sin 0) = 6$$ □



② $z = f(x,y)$. $x = g(s,t)$ $y = h(s,t)$.

Let G be the map $(s,t) \to (x,y) = (g(s,t), h(s,t))$.

$$J(G) = \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix} \begin{array}{l} \leftarrow \nabla x \\ \leftarrow \nabla y \end{array}$$

$$Jf = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right),$$

$$\text{so } \underline{J(f\circ G)} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix}$$

$$\Rightarrow \quad \frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t},$$

Example $z = e^x \sin y$ $x = st^2$, $y = s^2 t$,

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= (e^x \sin y)(t^2) + (e^x \cos y)(2st) \rightarrow (s,t)$$

$$= t^2 e^{st^2} \sin(s^2 t) + 2st\, e^{st^2} \cos(s^2 t).$$

$$\frac{\partial z}{\partial t} = (e^x \sin y)(2st) + (e^x \cos y)(s^2)$$

$$= 2st\, e^{st^2} \sin(s^2 t) + s^2 e^{st^2} \cos(s^2 t).$$

$\square$

$y=f(x)$ → derivative ⟷ product of ~

composition

Jacobian of composition ⟷ product of Jacobians

③ General case:

$$F = (f_1(y_1 \cdots y_n), \cdots f_m(y_1 \cdots y_n)),$$
$$G = (g_1(x_1 \cdots x_r) \cdots g_n(x_1 \cdots x_r))$$

memory

$$\Rightarrow \quad \boxed{\frac{\partial f_i}{\partial x_j} = \sum_k \frac{\partial f_i}{\partial y_k} \cdot \frac{\partial y_k}{\partial x_j},}$$

← (i,j)-th entry of $J(F \circ G)$.

$$J(F \circ G) = \left(\frac{\partial f_i}{\partial x_j}\right)$$

sum over all possible intermediate variable

In particular, if $f : \mathbb{R}^n \to \mathbb{R}$,

$$\frac{\partial f}{\partial x_j} = \sum_k \frac{\partial f}{\partial y_k} \cdot \frac{\partial y_k}{\partial x_j}$$

Example $u = x^4 y + y^2 z^3$, $x = rse^t$ $y = rs^2 e^{-t}$, $z = r^2 s \sin t$.

(r, s

$$(r, s, t) \longrightarrow (x, y, z) \rightarrow u.$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial s}.$$

$$= 4x^3 y \cdot re^t + (x^4 + 2yz^3)(2rse^{-t})$$
$$+ (3y^2 z^2)(r^2 \sin t),$$

when $(r, s, t) = (2, 1, 0)$ $(x, y, z) = (2, 2, 0)$.

$$\left.\frac{\partial u}{\partial s}\right|_{(2,1,0)} = 192.$$

at l)

□

$$\text{smooth} \xrightarrow{\text{linearization}} \text{Jacobian}$$

$$\text{composition} \quad \text{s} \longrightarrow \text{compo -}$$

linear

matrix multiplication

Lecture 10.

The chain rule:

Let $F: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $G: \mathbb{R}^r \longrightarrow \mathbb{R}^n$.

$$\underbrace{\left(\frac{\partial f_i}{\partial x_j}\right)}$$

$$J(F \circ G)_{\vec{x}} = J(F)_{\vec{y}} \cdot J(G)_{\vec{x}}$$

compare with the classical chain rule:

$$(f \circ g)'_x = f'_y \cdot g'_x$$

derivatives	multiplication of numbers / functions
⇝	⇝
Jacobian matrices	multiplication of matrices with real number or function entries.

1. differentials

$$df = f'(x)dx$$

For a differentiable function of two variables, $z = f(x,y)$, we define the differentials dx and dy to be independent variables. Then the differential dz, also called the total differential, is defined by.

$$\boxed{dz = f_x(x,y)dx + f_y(x,y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy}$$

Sometimes the notation df is used in place of dz.

dz is a function of four variables: x, y, dx, dy.

If we take $dx = \Delta x = x - x_0$, $dy = \Delta y = y - y_0$, then the differential of z is

$$dz = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

The linear approximation can be written as.

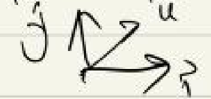
$$f(x,y) \approx f(x_0, y_0) + \underset{\sim}{dz}$$

Example $z = f(x,y) = x^2 + 3xy - y^2$.

$$dz = \underset{\sim}{\frac{\partial z}{\partial x}} \underline{dx} + \underset{\sim}{\frac{\partial z}{\partial y}} \underline{dy}$$

$$= (2x + 3y)dx + (3x - 2y)dy$$

4 variables.



2, Directional derivatives.

$\underset{\sim}{x = a}$

$\underline{y = b}$

$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}.$$

$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

The partial derivatives represent the rates of change of z in the x- and y- directions, that is, in the directions of the unit vectors $\vec{i}$ and $\vec{j}$.

Now let $\vec{u} = \langle a, b \rangle$ be a unit vector, we want to find the rate of change of z in this direction,

Definition The directional derivative of f at (x_0, y_0) in the direction of a unit vector $\vec{u} = \langle a, b \rangle$ is.

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha,\ y_0 + hb) - f(x_0, y_0)}{h}.$$

if this exists.

h scales

How to compute. $D_{\vec{u}} f(x_0, y_0)$?

We define a function g of the single variable h by

$$g(h) = f(x_0 + ha,\ y_0 + hb).$$

then $D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = g'(0).$

$$\begin{tikzcd} h \arrow[r] & (x,y) \arrow[r] & f(x,y) \\ \mathbb{R} \arrow[r] & \mathbb{R}^2 \arrow[r] & \mathbb{R}. \end{tikzcd}$$

on the other hand,

$g(h) = f(x, y)$, $x = x_0 + ha$, $y = y_0 + hb$.

So the chain rule gives

$$\begin{tikzcd} \mathbb{R}^1 \arrow[r, "{(x,y)}"] & \mathbb{R}^2 \arrow[r, "f"] & \mathbb{R}, \\ h \arrow[rr, bend right, "g"'] & (x,y) & {} \end{tikzcd}$$

$$g'(h) = \frac{\partial f}{\partial x} \frac{dx}{dh} + \frac{\partial f}{\partial y} \frac{dy}{dh}$$

$$= f_x(x,y)\, a + f_y(x,y)\, b,$$

$\nabla f = (f_x, f_y) \cdot \vec{u}$

Put $h = 0$, $(x, y) = (x_0, y_0)$ So.

$$\boxed{D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0)\, a + f_y(x_0, y_0)\, b}$$

Write $\vec{u} = \langle \cos\theta, \sin\theta \rangle$,

$$D_{\vec{u}} f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta .$$

$$= \underset{\text{vector}}{\nabla f} \cdot \underset{\text{vector}}{\vec{u}} .$$ ← number at a point

Remark $f_x(x,y)$ and $f_y(x,y)$. are enough to compute all directional derivatives. The rate of change in two directions control the rate of change in all directions.

$\vec{i}, \vec{j}$

Example $f(x,y) = x^3 - 3xy + 4y^2$ $\quad \vec{u} = (\cos\frac{\pi}{6}, \sin\frac{\pi}{6})$. ← unit vec

$$D_{\vec{u}} f(x,y) = f_x(x,y)\cos\frac{\pi}{6} + f_y(x,y)\sin\frac{\pi}{6}$$

$$= (3x^2 - 3y)\frac{\sqrt{3}}{2} + (-3x + 8y)\cdot\frac{1}{2}.$$

$$D_{\vec{u}}(1,2) = \frac{13 - 3\sqrt{3}}{2}.$$

Remark The definition and computation of directional derivatives are easily generalized to functions of three variables.

$$D_{\vec{u}} f(x_0,y_0,z_0) = \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb, z_0+hc) - f(x_0,y_0,z_0)}{h}$$

$\vec{u} = \langle a,b,c \rangle$ a unit vector.

$$D_{\vec{u}} f(x_0,y_0,z_0) = \nabla f \cdot \vec{u} .$$

$$\vec{u} \longrightarrow D_{\vec{u}} f(\vec{x})$$

$$S^1/S^2 \longrightarrow \mathbb{R}$$

3. Maximizing the directional derivative.

Suppose f is a function of two or three variables,

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = |\nabla f||\vec{u}| \cos\theta = |\nabla f| \cos\theta.$$

$|\vec{u}|=1$

The maximum value of $D_{\vec{u}} f$ is $|\nabla f|$ and it occurs when $\vec{u}$ has the same direction as $\nabla f(\vec{x})$.

↖ fix a (x, y) (x, y, z)

Remark, As $\vec{x}$ varies, $\vec{u}$ also varies.

$\nabla f(\vec{x})$ is fixed

$\vec{u}$ varies

$D_{\vec{u}} f$

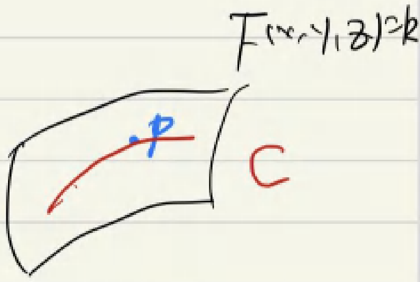
4. Tangent planes to level surfaces.

Suppose S is a surface with equation,

$$F(x, y, z) = k,$$

that is, it is a level surface of a function F of three variables.

Let $P(x_0, y_0, z_0)$ be a point on S. Let C be any curve that lies on the surface and passes through the point P,



We may write $C: \vec{r}(t) = (x(t), y(t), z(t))$, and $P = \vec{r}(t_0)$,

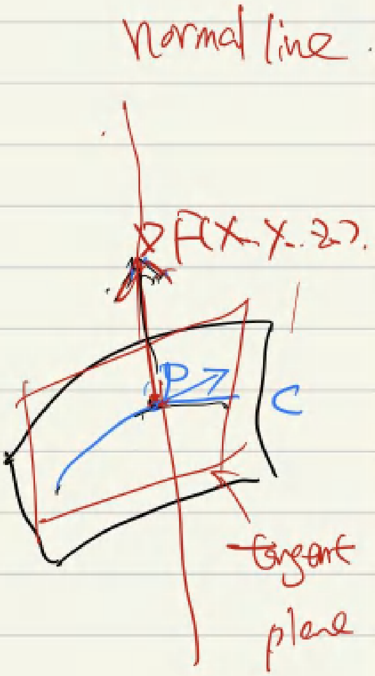
C lies on S

$$\Rightarrow \quad F(x(t), y(t), z(t)) = k,$$

$$\begin{tikzcd} t \arrow[r] & (x, y, z) \arrow[r] & F(x,y,z) \\ \mathbb{R} \arrow[r] \arrow[rr, bend right] & \mathbb{R}^3 \arrow[r] & \mathbb{R} \end{tikzcd}$$

$$\Rightarrow \quad \frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0.$$

$$= \nabla F \cdot \vec{r}'(t) = 0$$



$\nabla F \cdot \vec{r}'(t) = 0.$ ∀t

In particular.

$$\nabla F(x_0, y_0, z_0) \cdot \vec{r}'(t_0) = 0 \quad t_0$$

$\nabla F(x_0, y_0, z_0)$ is perpendicular to the tangent vector $\vec{r}'(t_0)$ to any curve C. on S that passes through P.

If $\nabla F(x_0, y_0, z_0) \neq \vec{0}$, the tangent plane to the level surface $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ is defined to be the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$. Its equation is

$$\boxed{\begin{aligned} F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) \\ + F_z(x_0, y_0, z_0)(z - z_0) = 0 \end{aligned}}$$

The normal line to S at P is the line passing through P and perpendicular to the tangent plane. The direction of the normal line is $\nabla F(x_0, y_0, z_0)$.

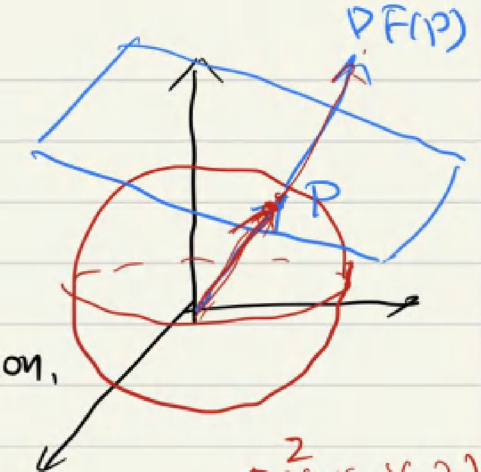
Example. $x^2 + y^2 + z^2 = 1. = S$ sphere.

$F = x^2 + y^2 + z^2$, $\nabla F = (2x, 2y, 2z)$.

$\in S$

At any point $P = (x_0, y_0, z_0)$. $\nabla F = 2(x_0, y_0, z_0)$, parallel to the position vector (x_0, y_0, z_0)

7,



This means that the tangent plane at P is normal to the vector $\overrightarrow{OP}$.

This agrees with our geometric intuition.

The equation of the tangent plane: $\nabla F \overset{2}{=} (2x_0, y_0, z_0)$

$$\underline{x_0}(x-\underline{x_0}) + \underline{y_0}(y-\underline{y_0}) + \underline{z_0}(z-\underline{z_0}) = 0$$

$$\Rightarrow \quad x_0x + y_0y + z_0z - 1 = 0. \quad 1 = x_0^2 + y_0^2 + z_0^2$$

The normal line passes through the origin. (just the line passing through O and P)

$$\langle x = x_0t, \quad y = y_0t, \quad z = z_0t \rangle.$$

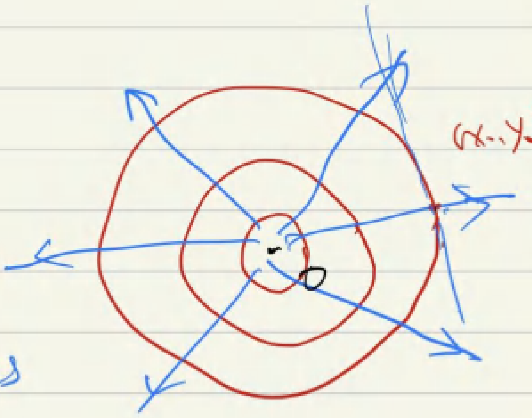
Gradient vectors of functions of three variables are perpendicular to tangent planes of level surfaces.

Gradient vectors of functions of two variables are perpendicular to the lever curves $f(x,y) = k$.

Example. $f(x,y) = x^2+y^2$

lever curves: circles.

$\nabla F(x_0, y_0) = 2(x_0, y_0)$.



∇f ⟷ level curves

"dual"

$(x,y,z) \to (f_x(x,y,z), f_y(x,y,z), f_z(x,y,z))$

Remark. $DF(x,y,z) = (f_x(x,y,z), f_y(x,y,z), f_z(x,y,z))$ ☆

assigns a vector to each point (x,y,z),

Thus defines a $\underset{\text{vector}}{\wedge}$ field.

∇: f
vector field.

5 Implicit differentiation.

We suppose that an equation of the form $F(x,y)=0$ defines y implicitly as a differentiable function of x, that is $y=f(x)$ where $F(x,f(x))=0$. for all x in the domain of f. We also assume F is differentiable,

do not solve equation

Apply chain rule to the equation

$$F(x,y)=0,$$

we obtain.

$$\frac{\partial F}{\partial x}\cdot \overset{=1}{\frac{dx}{dx}} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx} = 0.$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}} \quad ☆$$

we compute y' without solving $y=f(x)$ explicit formula

Example. $x^3+y^3=6xy.$

$F(x,y) = x^3+y^3-6xy$

$F_x = 3x^2-6y, \qquad F_y = 3y^2-6x.$

$$\Rightarrow y' = -\frac{3x^2-6y}{3y^2-6x} = -\frac{x^2-2y}{y^2-2x} \leftarrow f(x)? \qquad \square$$

So we find y' without writing $y=f(x)$ explicitly, Now we suppose that z is given implicitly as a function $z = f(x,y)$. by an equation of the form

$$F(x, y, z) = 0.$$

Apply chain rule:

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x} = 0.$$

But $\frac{\partial x}{\partial x} = 1 \quad \frac{\partial y}{\partial x} = 0.$ $\longleftarrow$

$x, y.$

$\frac{\partial x}{\partial x} = 1 \quad \frac{\partial}{\partial x}(y) = 0$

$$\Rightarrow \quad \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0.$$

$$\boxed{\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}}$$

Example $\quad x^3 + y^3 + z^3 + 6xyz = 1$

$F(x,y,z) = x^3 + y^3 + z^3 - 6xyz - 1 = 0.$

(x, y), z is function of (x,y)

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

$$\frac{\partial z}{\partial y} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

$\longleftrightarrow$ (x,y) symmetric ---

6. Implicit Function Theorem.

① Suppose all functions are "good enough", e.g. smooth, derivative

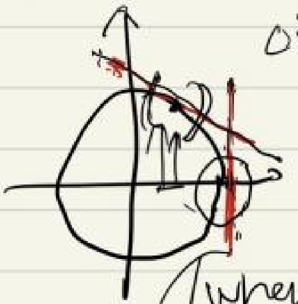
$F(x,y) = 0.$

when y is a function of x. ?

$y = kx+b$ $k \neq 0$

Example. $F(x,y) = x^2+y^2-1$

$\nabla F(x,y) = (2x, 2y).$



when $y \neq 0$ NOT a function

$x = 1.$

NOT the graph of a func $y = f(x)$

y is a function of $x \iff y \neq 0$
$\iff F_y \neq 0.$

Principle: local picture of smooth functions
$\updownarrow$
tangent plane at one point P.

Implicit Function theorem:

y is a function of x near some point P.

$\iff$ tangent line at P is not vertical.

(that is, tangent line is a graph).

$\iff$ $F_y \neq 0$. $(\nabla F \cdot L = 0)$. $\nabla F = (F_x, F_y)$

② $F(x,y,z)=0$.

z is a function of (x,y) near a point P,

$\Leftrightarrow$ tangent plane at P is a graph of some function

cannot vertical to xy plane.

$\Leftrightarrow$ $F_z(P)\neq 0$.

③ Generally, given

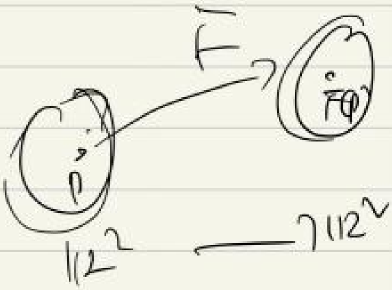
$$\begin{cases} f_1(x_1\cdots x_m)=0 \\ \\ f_n(x_1\cdots x_m)=0, \end{cases}$$

when $x_{n+1}\cdots x_m$ are functions of $x_1\cdots x_n$?

Implicit function theorem

$\Rightarrow$ check Jacobian. □

7, Inverse Function Theorem,



Let $F:\mathbb{R}^n\to\mathbb{R}^n$,

$F=(f_1(x_1\cdots x_n),\cdots,f_n(x_1\cdots x_n))$

Inverse function theorem

F establishes a bijection of neighborhoods of P and F(P),

$\Updownarrow$,

$\boxed{J(F)_P}$ is invertible $\Leftrightarrow$ $|J(F)_P|\neq 0$.

Example $\phi: (r,\theta) \to (x,y)$.

$x = r\cos\theta$

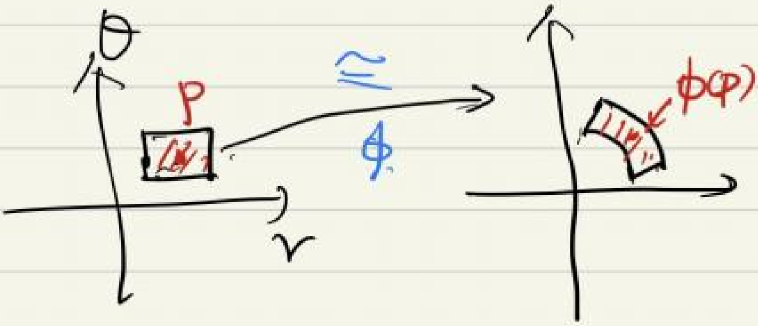
$y = r\sin\theta$.

$$|J(\phi)| = \left|\begin{pmatrix} \cos\theta, & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}\right| = r.$$

$$|J(\phi)| \neq 0 \iff r \neq 0.$$

This is just the polar coordinates.

when $r \neq 0$.
a neighbood of (r,θ)
is one-to-one to
a neighborhood of (x,y)

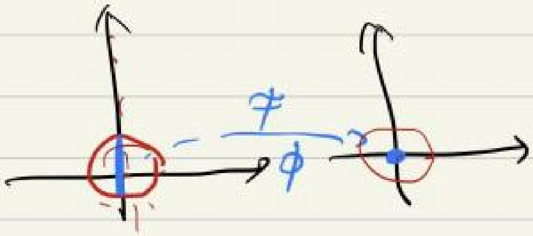


But when $r=0$,

ϕ maps the axis $(r=0)$ to the origin O. Collapsed

a neighborhood cannot one-to one
to a neighborhood of O;

$\phi^{-1}(O)$ is a curve.



double integrals.

$\downarrow \iint f \cdot dA = \int (\int f \, dy) \, dx$ /.

Lecture11. Extrama.

Last week

$F: \mathbb{R}^n \to \mathbb{R}^m$, $F = (f_1 \cdots f_m)$.

f'

$\underbrace{J(F) = \left(\frac{\partial f_i}{\partial x_j}\right)}$. $\rightsquigarrow$ $\underbrace{\text{chain rule}}$.

$F(x, y, z)$

$z = f(x, y)$

$\hat{=}$

(x, y, z)

Tangent spaces and tangent maps

$\Updownarrow$

Local properties of smooth functions.

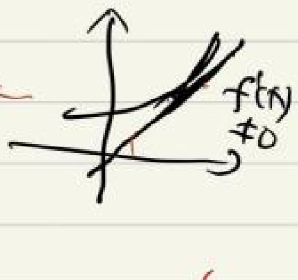
A

- ordinary derivatives $\rightsquigarrow$ find maximum and minimum values.
- partial derivatives $\rightsquigarrow$ locate maxima and minima of functions of $\boxed{\text{two variable}}$.
- More variables?

Linear algebra would be slightly more complicated
quadratic form theory.

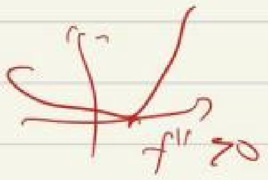
1. $y = f(x)$.

If f has a local maximum or minimum at x_0, $\Rightarrow$ $f'(x_0) = 0$ ←

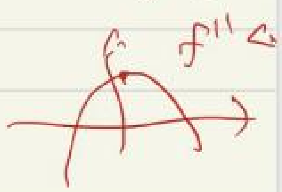


But we need $f''(x_0)$ to describe local properties of f near x_0:

① $f''(x_0) > 0$. local minimum.;



② $f''(x_0) < 0$ local maximum;



2,

← singular.

② $f''(x_0) = 0$

We cannot say anything

$f(x) = x^4$, ⟶ local minimum. $(x=0)$
$-x^4$ ⟶ local maximum $(x=0)$
x^3 ⟶ Not a local maximum or minimum $(x=0)$

$\left\{ f' = f'' = 0 \right.$

Functions of two variables?

$f(x,y)$ $= 0$?

$$0 = f'(x) \longrightarrow \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right),$$

$$> \; < \; f'' \longrightarrow H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

← Hessian matrix

← symmetric

Then: what is a "positive matrix"? ☆

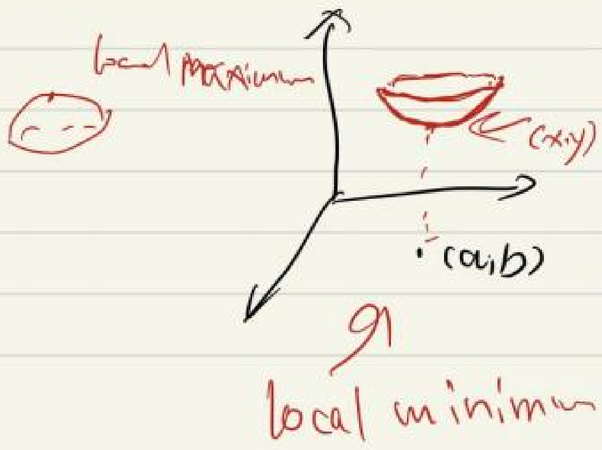
$f(x,y)$

"local" $f(a,b)$ →

2,

Definition A funcetion f of two variables has a local maximum at (a,b) if $f(x,y) \leq f(a,b)$. [$f(x,y) \leq f(a,b)$ for all points (x,y) in some disk with center (a,b)]. The number $f(a,b)$ is called a local maximum value.

Local minimum, ≥, local minimum value.



"global"

Definition If $f(x,y) \le f(a,b)$ for all points (x,y) in the domain of f, then f has an absolute maximum at (a,b).

$\geq$ minimum.

Theorem If f has a local maximum or minimum at (a,b) and the first order derivatives exist there, then

$$f_x(a,b) = f_y(a,b) = 0.$$

Proof: Consider the functions:

$$g(x) = f(x,b), \qquad h(y) = f(a,y)$$

$g(x)$ has a local maximum or minimum, so.

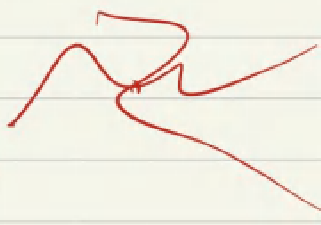
$$g'(a) = f_x(a,b) = 0$$ by definition □

Geometrically, the tangent plane at (a,b) is horizontal.

Definition A point (a,b) is called a critical point of f if $\nabla f = \vec{0}$, or one of the partial derivatives does not exist.

$(f_x(a,b) = f_y(a,b) = 0)$ ⟶ solve equations

Example $f(x,y) = x^2 + y^2 - 2x - 6y + 14$.



① $f_x = 2x - 2$, $f_y = 2y - 6$. ← $x = 1$, $y = 3$

So the only critical point is $(1,3)$.

② $f(x,y) = (x-1)^2 + (y-3)^2 + 4$. ← complete square.

$(1,3)$ is a local minimum, actually an absolute minimum.

$(x,y) \neq (1,3)$

$f(x,y) > 4 = f(1,3)$

$\mathbb{R}^m$

3, Some linear algebras (on $\mathbb{R}^2$)

- quadratic forms; symmetric matrices.

A <u>quadratic form</u> is polynomial with terms all of degree two.

$\underline{x^2} \longrightarrow 2$

Example $4x^2+2xy-3y^2$. $\quad \underline{xy} \longrightarrow 2 \quad \underline{y^2} \longrightarrow 2$

A 2x2 matrix $\begin{pmatrix} a & b \\ \underline{c} & d \end{pmatrix}$ is called symmetric if

$b=c$,

Example $\begin{pmatrix} 4 & 1 \\ 1 & -3 \end{pmatrix}$ is symmetric, $\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$ is <u>NOT symmetric</u>

There is a one-to-one correspondence between.

quadratic forms $\qquad$ 2x2 $\boxed{\text{symmetric}}$ matrices.

$$ax^2+\underline{2b}xy+cy^2 \longleftrightarrow \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Remark $$ax^2+\underline{2b}xy+cy^2 = (x,y)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \Leftarrow$$ matrix multiplication

Example $$4x^2+\underline{2}xy-3y^2 = (x,y)\begin{pmatrix} 4 & \underline{1} \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

□

naive $\begin{pmatrix} a & b \\ c & d \end{pmatrix} > 0 \iff a, b, c, d > 0$

NOT a good def. —

X

Definition. A symmetric matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is called positive definite,

iff, $ax^2+2bxy+cy^2>0 \quad \forall\ (x,y)\neq\vec{0}$,

- - - negative definite

- - - $<0. \quad \forall (x,y)\neq\vec{0}$

indefinite.

takes on both positive and negative values.

$$ax^2+2bxy+cy^2$$
$$= a(x+\frac{b}{a}y)^2+(c-\frac{b^2}{a})y^2.$$
$$= a(x+\frac{b}{a}y)^2+\boxed{\frac{D}{a}}y^2 \quad (a\neq 0). \quad \leftarrow D=\det\begin{vmatrix} a & b \\ b & c \end{vmatrix}$$
$$= ac-b^2.$$

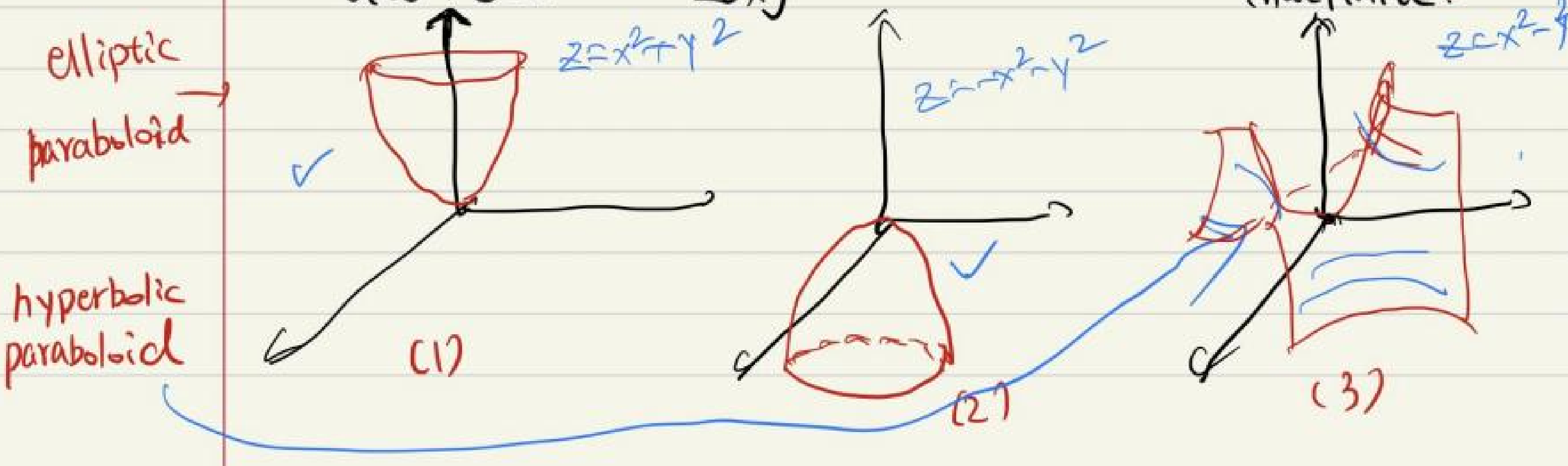
(1) $a>0.\ D>0. \quad \Rightarrow$ positive definite $\quad \Big\}\ D>0 \Rightarrow a\neq 0$

(2) $a<0 \quad D>0 \quad \Rightarrow$ negative definite.

(3) $a\neq 0 \quad D<0 \quad \Rightarrow$ indefinite.

if $a=0.,\ c\neq 0, \quad 2bxy+cy^2=c(y+\frac{bx}{c})^2-\frac{b^2}{c}x^2$; indefinite.

$a=0.\ c=0 \quad 2bxy \quad$ indefinite.



↑ change of coordinate.

Remark Up to change of coordinates, any positive definite (negative, definite, indefinite) quadratic form can be written as x^2+y^2 $(-x^2-y^2,\ x^2-y^2)$. (linear algebra)

Remark. when $D=0$, $ax^2+2bxy+cy^2 \sim cy^2$, Its graph is a cylinder. ⊗

4. Second derivative test

Let $P=(x_0, y_0)$, $x=(x,y)$,

~ at p.

function of (x,y)

The second degree Taylor polynomial is:

constant matrix.

$$T_2(x) = \underbrace{f(p)}_{\text{constant}} + \underbrace{\overset{\text{constant vector}}{\boxed{Df(p)}}\cdot(x-P)}_{\text{linear}} + \underbrace{\frac{1}{2}(x-p)\boxed{H(f)(p)}(x-p)^t}_{\text{quadratic form}}.$$

Remark. $f(x) = T_2(x) + \underline{\text{higher order terms}},$ ⇐ negligible near p.

If P is a critical point, $Df(p)=0$,

when x is sufficiently close to P, the behavior of $f(x)$ is dominated by $(x-P)H(f)(p)(x-P)^t$.

(x,y)~P.

near p. $f \sim f(p)$ + quadratic form.

$H(f)(p)$ positive definite ⇒ local minimum.

$H(f)(p)$ negative definite ⇒ local maximum

$H(f)(p)$ indefinite ⇒ not a local maximum or minimum.

Second Derivative Test

Suppose the second partial derivatives of f are continuous on a disk with center (a,b), and suppose that $f_x(a,b) = f_y(a,b) = 0$. Let

$$D = f_{xx}(a,b)\, f_{yy}(a,b) - f_{xy}(a,b)^2$$

(a) If $D>0$ and $f_{xx}(a,b)>0$, then $f(a,b)$ is a local minimum;
(b) If $D>0$ and $f_{xx}(a,b)<0$, then $f(a,b)$ is a local maximum;
(c) If $D<0$. then $f(a,b)$ is not a local maximum or minimum.

↑ saddle point

Remark If $D=0$, the test gives no information.

$f''(x_0)=0$

Example $f(x,y) = x^4+y^4-4xy+1$

1) critical points,

$f_x = 4x^3-4y \quad f_y = 4y^3-4x.$

$f_x = f_y = 0.$

$\Rightarrow \underline{0 = x^9-x} = x(x^2-1)(x^2+1)(x^4+1)$

$x=0,1,-1 \Rightarrow y=x^3=0,1,-1$

$f_x=0 \Rightarrow \underline{y=x^3.}$

$f_y=0 \Rightarrow x^9-x=0$

The three critical points are $(0,0)$, $(1,1)$ $(-1,-1)$,

2) $H(f) = \begin{pmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{pmatrix}$

$(0,0)$, : $D=-16.$ saddle point;

$(1,1)$: $D(1,1)=128$, $f_{xx}(1,1)=12$, local minimum.

$(-1,-1)$: local minimum.

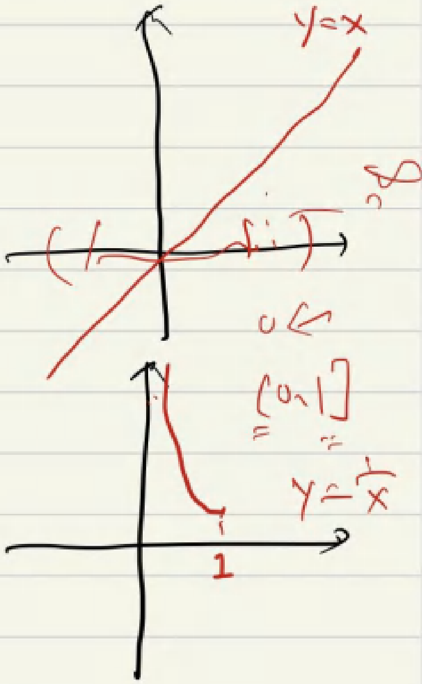
f' f'':

$J(f)$ (∇f). $H(f)$. [local ⇒ local maximum or minimum] 8.

5 Some topology.

Extreme value Theorem;

Let f be a continuous of one variable. on a closed ① ② interval [a,b]. Then f has an absolute minimum value and an absolute maximal value.



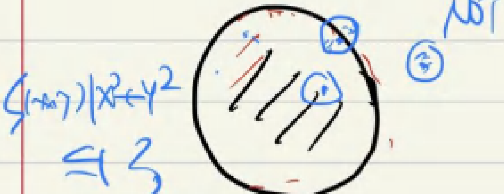
1) [a,b] is **bounded**; $a \neq -\infty$. $b \neq \infty$:

2). [a,b] is closed;

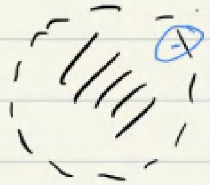
A bounded set in $\mathbb{R}^2$ is one that is contained within some disk.

A closed set in $\mathbb{R}^2$ is one that contains all its boundary points. [A boundary point of D is a point (a,b) such that every disk with center (a,b) contains points in D and also points not in D],

$\{(x,y) \mid x^2+y^2<1\}$. $\overset{\uparrow}{\partial D}$.



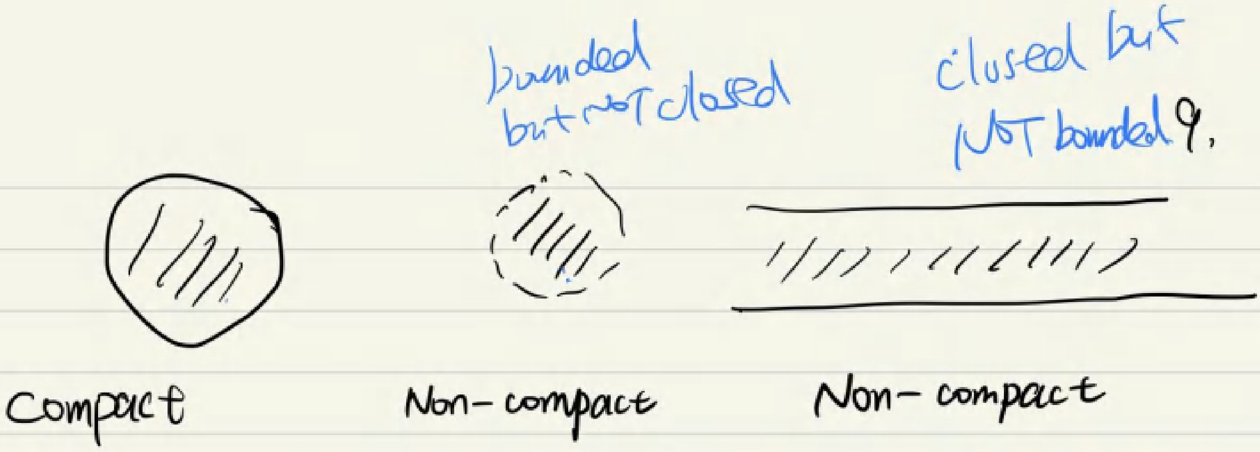
closed



NOT closed

NOT closed

A compact set in $\mathbb{R}^2$ is a bounded closed set.



closed interval ⟷ compact subsets

6. Extreme value Theorem for Functions of Two variables.

If f is continuous on a compact set D on $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D. ⇒ existence - - -

To find the absolute maximum and minimum value of a continuous function on a compact set D:

☆ 1. Find the values at the critical points of f in D; in the interior of D

2. Find the extreme value of f on the boundary of D.

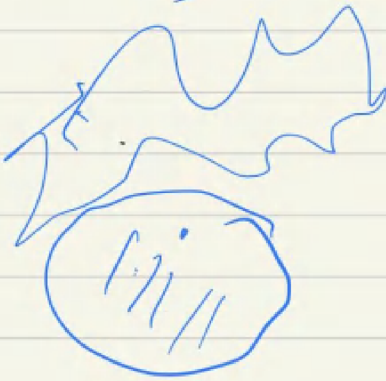
3. compare 1 & 2.

Example $f(x,y) = x^2 - 2xy + 2y$, $D = \{(x,y) \mid 0 \le x \le 3,\ 0 \le y \le 2\}$. ← rectangle.

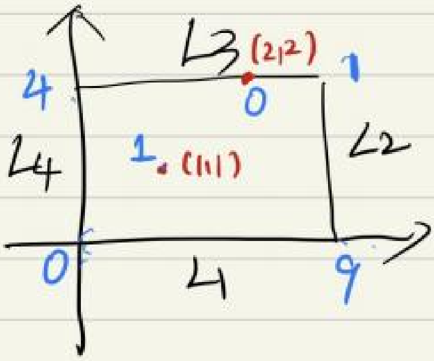
(1) $f_x = 2x - 2y$ $f_y = -2x + 2$,
$f_x = f_y = 0$

$\Rightarrow$ critical point $(1,1)$.

$f(1,1) = 1$.



10.



(2).

on L_1:

$f(x,0) = x^2$. $\quad 0 \leq x \leq 3$

minimum value: $f(0,0) = 0$.

maximal value: $f(3,0) = 9$.

on L_2:

$f(3,y) = 9-4y \quad 0 \leq y \leq 2$

minimum value: $f(3,2) = 1$

maximum value: $f(3,0) = 9$

1. value at critical.

①

on L_3

$f(x,2) = x^2-4x+4 \quad 0 \leq x \leq 3$.

minimum value: $f(2,2) = 0$.

maximum value: $f(0,2) = 4$.

2. maximum ∂D

⑨

on L_4:

$f(0,y) = 2y \quad 0 \leq y \leq 2$.

minimum ∂D

⓪

minimum value: $f(0,0) = 0$

maximum value: $f(0,2) = 4$

(3) absolute minimum value $= 0. = f(0,0) = f(2,2)$.

absolute maximum value $= 9 = f(3,0)$.

How to find the extreme values of f on the boundary?

1. parameterize the boundary, the $f|_{\partial D}$ is a function of one variable.

Example $D = \{(x,y) \mid x^2+y^2 \le 1\}$, $\partial D = \{(x,y) \mid x^2+y^2 = 1\}$.

$= (x,y)$

$= (\cos\theta, \sin\theta)$

$f|_{\partial D} = f(\cos\theta, \sin\theta) = F(\theta)$.

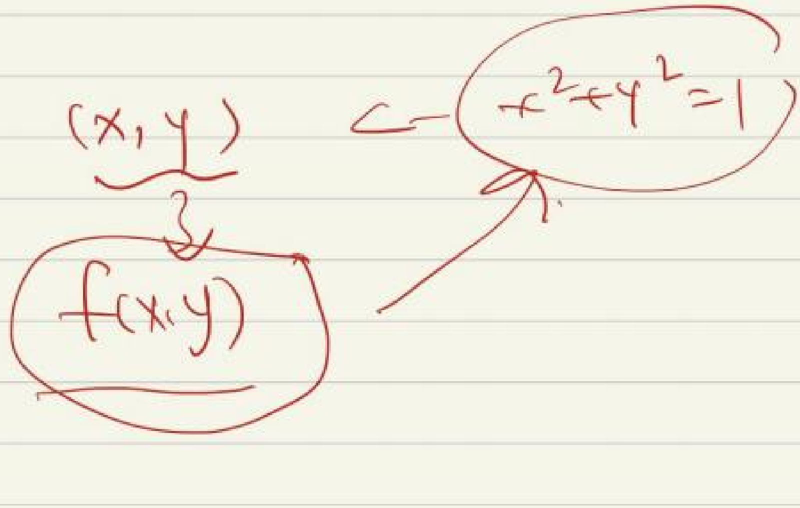
F' . . .

② Lagrange multipliers.

Lecture 12. ☆

$t \to \partial D$,

$t - - \vec{r}(t)$.



Lecture 12. Lagrange Multipliers

L11: critical points, $Df = \vec{0}$

second derivatives test: $\Big\}$ ⇒ local maximum/minimum.

$H(f) > 0$. positive definite $(f_{xx} > 0.\ D > 0)$

< 0. negative definite. $(f_{xx} > 0\ D > 0)$

$>< 0$? indefinite. $(D < 0)$

$\underline{f(x,y)} \leftarrow$ No relations

Goal: maximize or minimize a general function $f(x,y,z)$.

subject to a constraint (or a side condition) of the form $g(x,y,z) = k$.

1. Naive idea: solve $z = Z(x,y)$ from the constraint, then $f = f(x,y,z) = F(x,y)$ is a function of two free variables, (x,y,z)

Example. Find the maximum of $V = xyz$, where.

$$2xz + 2yz + xy = 12. \qquad x, y, z > 0$$

Solution From the equation

$$2xz + 2yz + xy = 12,$$

$$\Rightarrow \quad z = \frac{12 - xy}{2(x+y)}$$

$$\text{so } V = xyz = xy \cdot \frac{12 - xy}{2(x+y)} = \frac{12xy - x^2y^2}{2(x+y)}$$

$$V = \frac{12xy - x^2y^2}{2(x+y)}.$$ ← more complicated

$$\underbrace{\frac{\partial V}{\partial x} = \frac{y^2(12-2xy-x^2)}{2(x+y)^2}}\qquad \frac{\partial V}{\partial y} = \frac{x^2(12-2xy-y^2)}{2(x+y)^2}$$

interchange $x \leftrightarrow y$

critical points: $\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0$ $\quad x, y > 0.$

$$\Rightarrow x^2 = y^2 \Rightarrow x = y$$

$$x = y = 2, \quad z = 1, \quad V = 4$$

Then check that $(2,2)$ is a local maximum. ☐

However, it is not easy to find an explicit formula

$z = z(x, y)$ from the constraint $g(x, y, z) = 0$.

$x^2+y^3+z^3$
$+xyz=4$

z^3+xyz
$+x^2+y^3 = 0$

$\downarrow$

$\underbrace{z(x,y)}$

Lagrange multiplier is a useful method to solve this problem

2, Functions of two variables.

Find the extreme values of $f(x,y)$ subject to a constraint of the form $g(x,y) = k$.

$\Leftrightarrow$

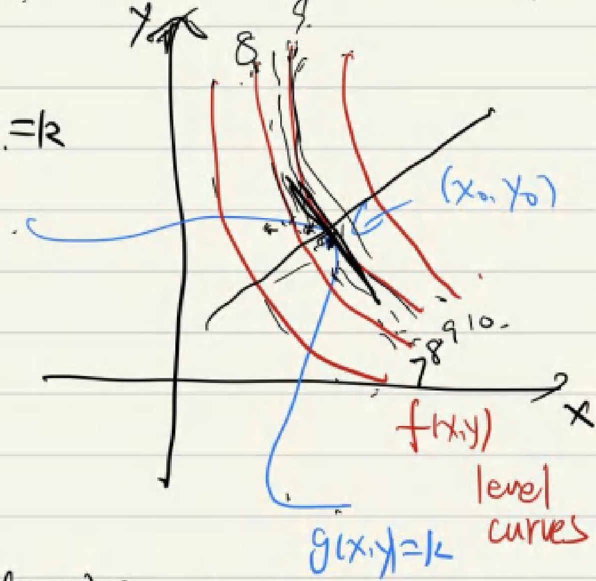
We seek the extreme values of $f(x,y)$ when the point (x,y) is restricted to lie on the level curve $g(x,y) = k$

can be

when $g(x,y)=k$, $f(x,y)=7, 8, 9,$

$f(x,y) \neq 10$

3,



To maximize $f(x,y)$ subject to $g(x,y)=k$ is to find the largest value of c such that the level curve $f(x,y)=c$ intersects $g(x,y)=k$.

This happens when these curves just touch each other,

① $g(x,y)=k$ $f(x,y)=c$.

$\Leftrightarrow$

when they have a common tangent line.

$\Leftrightarrow$

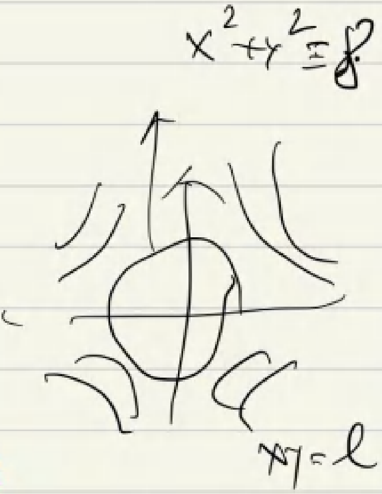
the normal lines at the point where they touch are identical.

$\Leftrightarrow$

gradient vectors are parallel:

$$\nabla f(x_0,y_0) = \lambda \nabla g(x_0,y_0) \quad ☆$$

for some scalar λ.



Example: $f(x,y)=xy$ $g(x,y)=x^2+y^2=8$

$\nabla f=(y,x)$ $\nabla g=(2x,2y)$.

$$\boxed{(y,x)=\lambda(2x,2y)}$$ $x^2+y^2=8$

$2y^2=2x^2 \Rightarrow x^2=y^2$ $x^2=4$ $y^2=4$ } 3 functions 3 variables

← 4 points

$(x,y)=(\pm 2, \pm 2)$

$f(x,y)=\pm 4$ ← maximum/minimum.

$y=2\lambda x$

$x=2\lambda y$

$\lambda(2x^2-2y^2)=0$ ← $y\cdot(2\lambda y)=x\cdot(2\lambda x)$

3. Functions of three variables.

Find the extreme values of $f(x,y,z)$ subject to the constraint $g(x,y,z)=k$.

level curves $\longleftrightarrow$ level surfaces.

if the maximum value of f is $f(x_0,y_0,z_0)=c$, then the level surface $f(x,y,z)=c$ is tangent to the level surface $g(x,y,z)=k$. and so the gradient vectors are parallel.

non-singular surface

Therefore, if $\nabla g(x_0,y_0,z_0)\neq\vec{0}$, there is a number such that

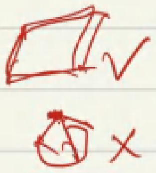
$$\boxed{\nabla f(x_0,y_0,z_0)=\lambda\nabla g(x_0,y_0,z_0)}$$

λ is called a Lagrange multiplier.

Method of Lagrange Multipliers

To find the maximum and minimum values of $f(x,y,z)$ subject to the constraint $g(x,y,z)=k$ [assuming that these extreme values exist and $\nabla g\neq\vec{0}$ on the surface $g(x,y,z)=k$]:

extreme value theorem



vertex of a cone

(a) Find all values of x,y,z, and λ, such that

$$\begin{cases}\nabla f(x,y,z)=\lambda\nabla g(x,y,z).\\ g(x,y,z)=k.\end{cases}$$

b) Evaluate f at all points (x,y,z) in (a). then compare these values.

Remark $f_x = \lambda g_x$ $f_y = \lambda g_y$, $f_z = \lambda g_z$. $g(x,y,z) = k$.

4 variables, 4 equations

$\Rightarrow$ solutions should be a $\underbrace{\text{discrete set}}$.

(if. the equations are "general")

Example $V = xyz$.

$g = 2xz + 2yz + xy = 12$. $\quad x, y, z > 0$.

Solution $\nabla V = (yz, xz, xy)$

$\nabla g = (2z+y, 2z+x, 2x+2y)$.

$$\Rightarrow \begin{cases} yz = \lambda(2z+y) & \text{deg 2} & ① & V_x = \lambda g_x \\ xz = \lambda(2z+x) & & ② & V_y = \lambda g_y \\ xy = \lambda(2x+2y) & & ③ & V_z = \lambda g_z \\ 2xz + 2yz + xy = 12 & & ④ & g = k \end{cases}$$

find some linear relations

Multiply ①, ②, ③ by x, y, z.

$$xyz = \lambda(2xz + xy) \quad \text{deg 3} \quad ⑤$$
$$xyz = \lambda(2yz + xy) \quad ⑥$$
$$xyz = \lambda(2xz + 2yz) \quad ⑦$$

by ①②③

$\lambda \neq 0$: otherwise, $xy = yz = xz = 0$, $g \neq 12$.

From ⑤⑥, $2xz = 2yz$ $\Rightarrow \boxed{x = y, = 2z}$ deg 1

From ⑥⑦ $y = 2z$

Put $x = y = 2z$ in ④ $\Rightarrow 12z^2 = 12$, $\Rightarrow z = 1$, $x = y = 2$,

$$\underbrace{2\cdot(2z)\cdot z + 2\cdot(2z)\cdot z + (2z)\cdot(2z)}_{12z^2}$$

$V(2,2,1) = 4$. □

Example2. $f(x,y) = x^2+2y^2$.
$g = x^2+y^2=1$.

Solution ① Lagrange multipliers.

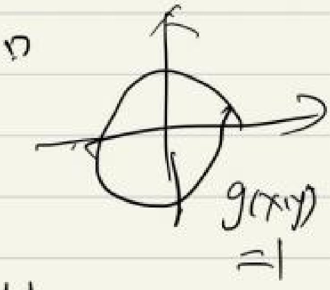
$\nabla f = (2x, 4y)$ $\quad$ $\nabla g = (2x, 2y)$.

$$\Rightarrow \begin{cases} 2x = 2\lambda x & ① \\ 4y = 2\lambda y & ② \\ x^2+y^2=1. & ③ \end{cases} \qquad x(1-\lambda)=0$$

From ①.
$$\begin{cases} \underline{x=0} \Rightarrow y=\pm 1 \\ \text{or } \underline{\lambda=1} \xrightarrow{②} y=0,\ x=\pm 1. \end{cases}$$

$(\pm1,0)$; $(0,\pm1)$ ← 4 points.

$f(0,1)=2$ $\quad$ $f(0,-1)=2$ $\quad$ $f(1,0)=1$ $\quad$ $f(-1,0)=1$
maximum value: 2.
minimum value: 1.



② on the curve $x^2+y^2=1$.

$$f(x,y) = x^2+2y^2 = (x^2+y^2)+y^2 = \underline{y^2+1}.$$

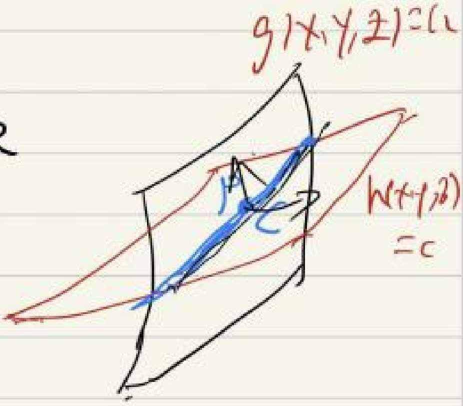
But now $\underline{-1 \le y \le 1}$. $\quad$ $(y \in \{(x,y) \mid x^2+y^2=1\})$

$\Rightarrow$ maximum $=2$,
minimum $=1$.

□

4. Find the maximum and minimum values of a function $f(x,y,z)$. subject to two constraints. of the form.
$g(x,y,z)=k$, and $h(x,y,z)=c$,

$\Leftrightarrow$ extreme values of f when (x,y,z) is restricted to lie on the curve ~~of~~ intersection C ~~of~~ the level surfaces. $g(x,y,z)=k$ and $h(x,y,z)=c$.



Suppose f has such an extreme value at a point $P = (x_0, y_0, z_0)$

∇f is orthogonal to c at P.

But c is orthogonal to ∇g, ∇h,

so ∇f is a linear combination of ∇g and ∇h:

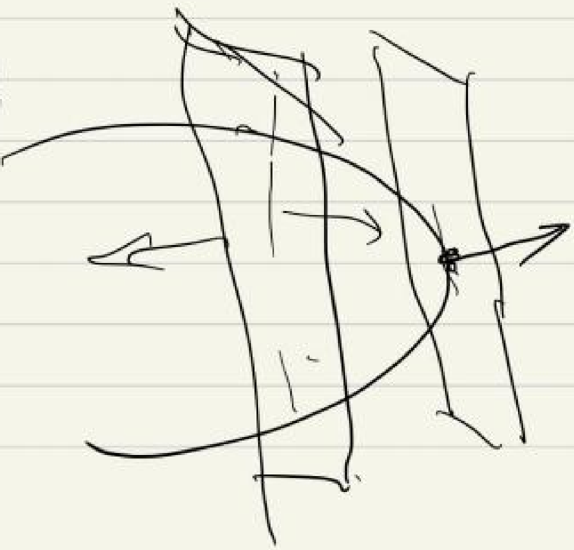
There are numbers λ and μ such that

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0).$$

$$\begin{cases} f_x = \lambda g_x + \mu h_x ; \cdots f_y = \lambda g_y + \mu h_y, \quad f_z = \lambda g_z + \mu h_z. \\ g(x,y,z) = k \qquad h(x,y,z) = c. \end{cases}$$

{5 variables x, y, z, λ, μ}

nabla. ∇
lambda λ.



Example $f(x,y,z) = x+2y+3z.$

$g(x,y,z) = x-y+z=1.$

$h(x,y,z) = x^2+y^2=1.$

Solution. $\nabla f = (1,2,3)$ $\quad \nabla g = (1,-1,1)$ $\quad \nabla h = (2x, 2y, 0).$

$$\begin{cases} 1 = \lambda + 2x\mu. & ① \\ 2 = -\lambda + 2y\mu. & ② \\ 3 = \lambda & ③ \\ x-y+z=1 & ④ \\ x^2+y^2=1 & ⑤ \end{cases}$$

$3=\lambda \overset{①}{\Rightarrow} x = -1/\mu.$ $\qquad 1 = 3+2x\mu \Rightarrow x = -\frac{1}{\mu}.$

$\overset{②}{\Rightarrow} y = \frac{5}{2\mu}.$

$\overset{⑤}{\Rightarrow} \quad \frac{1}{\mu^2} + \frac{25}{4\mu^2} = 1$

$\Rightarrow \mu^2 = \frac{29}{4}, \qquad \mu = \pm\frac{\sqrt{29}}{2}$

$x = \mp\frac{2}{\sqrt{29}}, \quad y = \pm\frac{5}{\sqrt{29}}, \quad z = 1 \pm \frac{7}{\sqrt{29}},$

The corresponding values of f are, $3 \pm \sqrt{29},$

maximum value: $3+\sqrt{29},$

minimum value: $3-\sqrt{29},$

Example. (Workbook, Problem 10)

$f(x,y,z) = x^2+y^2+z^2$ → minimal value

$g(x,y,z) = x^2y - z^2 + 9 = 0.$

Solution.

$\nabla f = (2x, 2y, 2z)$ $\nabla g = (2xy, x^2, -2z).$

$$\begin{cases} 2x = 2\lambda xy. & ① \\ 2y = \lambda x^2 & ② \\ 2z = -2\lambda z & ③ \\ x^2y - z^2 + 9 = 0 & ④ \end{cases}$$

From ③ $(\lambda+1)z = 0.$

if $z \neq 0.$ $\Rightarrow \lambda = -1,$

if $x=0.$ $\xRightarrow{②} y = 0 \xRightarrow{④} z = \pm 3$

$\boxed{(0.0, \pm 3)}$

if $x \neq 0$; ① $\Rightarrow y=-1$, ② $\Rightarrow x^2 = 2.$

$\xRightarrow{④} z^2 = 7$

$\boxed{(\pm\sqrt{2}, 0, \pm\sqrt{7})}$

if $z = 0$:

$$\begin{cases} x = \lambda x y & \textcircled{5} \\ 2y = \lambda x^2 & \textcircled{6} \\ x^2 y + 9 = 0 & \textcircled{7} \end{cases}$$

if $x=0 \overset{\textcircled{7}}{\Longrightarrow} 9=0$, contradiction.

so. $x \neq 0$

$\textcircled{5} \Rightarrow \lambda y = 1 \quad \textcircled{8}$

$\textcircled{6}\ \textcircled{8} \Longrightarrow$

$(2y^2 - x^2)\lambda = 0.$

but $\lambda \neq 0$ (otherwise $x=y=0$, $9=0$).

so $x^2 = 2y^2$.

$\overset{\textcircled{9}}{\Longrightarrow} \quad y = -\sqrt[3]{\frac{9}{2}} \qquad x = \pm\sqrt{2}\cdot\sqrt[3]{\frac{9}{2}}$

$$\boxed{\left(\pm\sqrt{2}\cdot\sqrt[3]{\frac{9}{2}},\ -\sqrt{\frac{9}{2}},\ 0\right)}$$

Compare the value of f at these special points,

we find that the minimal of f

is $f\left(\pm\sqrt{2}\sqrt[3]{\frac{9}{2}}, -\sqrt{\frac{9}{2}}, 0\right)$

$= 3y^2 = 3\cdot\left(\frac{9}{2}\right)^{2/3}$ □

Lecture 13

Double integrals over rectangles

1. Review of the definite integral.

If $f(x)$ is defined for $a \leq x \leq b$, we start by dividing the interval $[a,b]$ into n subintervals $[x_{i-1}\ x_i]$ of equal width $\Delta x = (b-a)/n$. and we choose sample points x_i^* in these subintervals. Then we form the Riemann sum

$$\sum_{i=1}^{n} f(x_i^*) \Delta x.$$

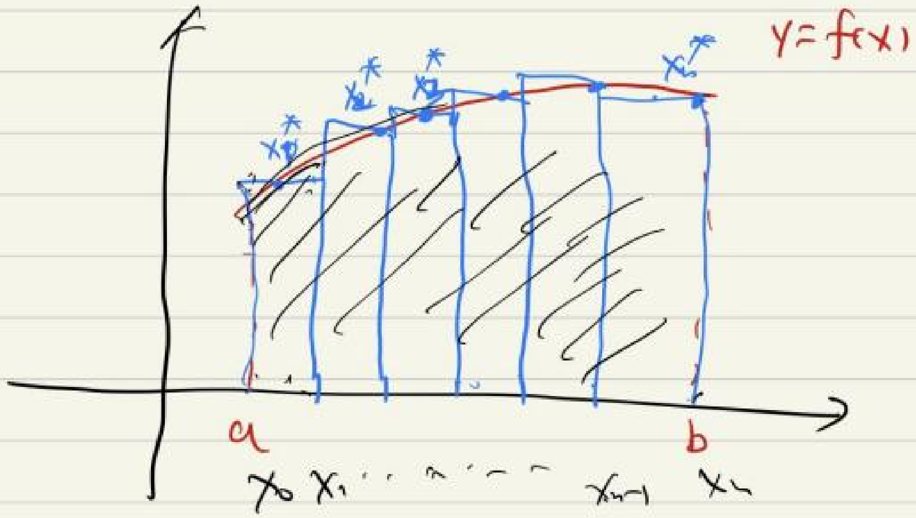
and take the limit of such sums as $n \to \infty$ to obtain the definite integral of f from a to b:

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x.$$

sample points arbitrary.

In the special case where $f(x) \geqslant 0$,

$\int_a^b f(x)dx$ represents the area under the curve $y=f(x)$ from a to b.



$$\left\{\begin{array}{l} \text{(Sub) interval} \longrightarrow \overset{\text{(sub)}}{\text{rectgles}} \\ \Delta x \longrightarrow \Delta A \end{array}\right.$$

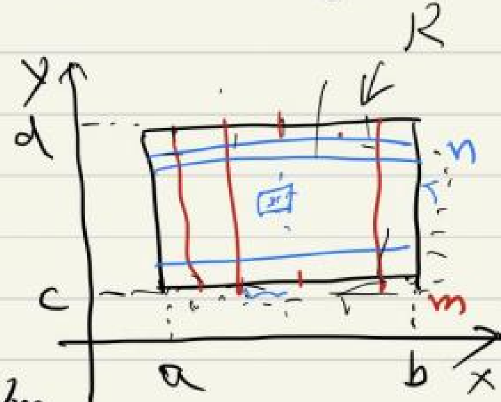
2,

2, Definition of double integrals

Cover rectangles).

f: function of two variables defined on a closed rectangle

$$R=[a,b]\times[c,d]=\{(x,y)\in\mathbb{R}^2 \mid a\le x\le b,\ c\le y\le d\}.$$



· divide the rectangles into subrectangles :

divide $[a,b]$ into m subintervals $[x_{i-1},x_i]$ of equal width $\Delta x=(b-a)/m$.

$[c,d]$ ··· n.

$[y_{j-1},y_j]$ ··· $\Delta y=(d-c)/n$.

consider the "product division".

mn rectangles $R_{ij}=[x_{i-1},x_i]\times[y_{j-1},y_j]$.

area $\Delta A=\Delta x\Delta y$.

any point in R_{ij}

Choose a sample point (x_{ij}^*, y_{ij}^*) in each R_{ij},

define the double Riemann sum:

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A=\sum_i\sum_j f(x_{ij}^*, y_{ij}^*)\Delta x\Delta y.$$

(mn) terms.

$y=f(x)$ $\int f(x) = \text{area}$

Definition The double integral of f over the rectangle R is.

$$\iint_R f(x,y)\,dA = \lim_{\substack{m\to\infty,\\ n\to\infty}} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

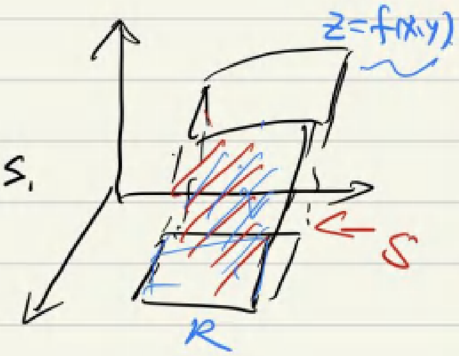
if this limit exists.

Geometric intepretation:

Suppose that $f(x,y) \geqslant 0$, The graph of f is a surface with equation $z=f(x,y)$. Let S be the solid that lies above R and under the graph of f.

$S = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \leq z \leq f(x,y), (x,y) \in R\}$.

$\iint_R f(x,y)\,dA$ is exactly the volume of S.



Remark: A function f is called integrable if the limit in the definition exists.

All integrable function over R is bounded. $|f| \leq M$

Theorem: If f is bounded over R, then

f is integrable over R if and only if the set of discontinuous points is a set of measure zero.

Lesbosgue measure

- bounded, continuous except on a finite number of smooth curves.
- continuous function (extreme value theorem).

area

The average value of a function f of two variables defined on a rectangle R is

$$f_{ave} = \frac{1}{A(R)} \iint_R f(x,y)\,dA.$$

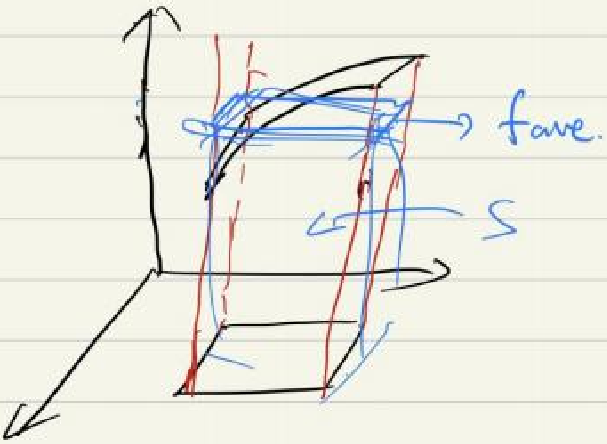
where $A(R)$ is the area of R.

If $a \le f(x,y) \le b$, $(x,y) \in \mathbb{R}$,
then
$$\iint_R f(x,y)\,dA \le \iint_R b\,dA = b\,A(R),$$

so $f_{ave} \le b$. similarly $a \le f_{ave}$.

In particular, if f is continuous, there exists $(x_0,y_0) \in R$,

such that $f_{ave} = f(x_0,y_0)$.



$$\iint_R f(x,y)\,dA = \text{volume of } S$$

$$\boxed{f_{ave}}\,A(R) = \text{volum of a "cube".}$$

function of y $\leftarrow \int_c^d f(x,y)dy \; \overline{x = \text{constant}}$

3. Suppose that f is a function of two variables that is integrable on the rectangle $R=[a,b]\times[c,d]$.

We use the notation $\int_c^d f(x,y)\,dy$ to mean that x is held fixed and f(x,y) is integrated with respect to y from $y=c$ to $y=d$. ← "partial integration"

Now $\int_c^d f(x,y)dy$ is a number that depends on the value of x, so it defines a function of x.

$x \to \mathbb{R}$

The integral

$$\int_a^b\int_c^d f(x,y)\,dy\,dx := \int_a^b\left[\underbrace{\int_c^d f(x,y)\,dy}\right]dx$$

$x \to \int_c^d f(x,y)dy \in \mathbb{R}$

is called an "iterated integral". function of x

x varies

Similarly

$$\int_c^d\int_a^b f(x,y)\,dx\,dy := \int_c^d\left[\int_a^b f(x,y)\,dx\right]dy,$$

← integration of one varible twice

Example $\int_0^3 \underline{\int_1^2 x^2y\,dy}\,dx$: → function of x, x fixed

$$\int_1^2 x^2y\,dy = \underline{x^2}\int_1^2 y\,dy = \underline{x^2}\cdot\frac{y^2}{2}\Big|_1^2 = \underline{\frac{3}{2}x^2}.$$

function of x

$$\int_0^3\int_1^2 x^2y\,dydx = \int_0^3 \frac{3}{2}x^2dx = \frac{x^3}{2}\Big|_0^3 = \frac{27}{2}.$$

Example $\int_1^2\int_0^3 x^2y\,dx\,dy$:

$$\int_0^3 x^2\underline{y}\,dx = \underline{y}\cdot\int_0^3 x^2dx = y\cdot\frac{x^3}{3}\Big|_0^3 = 9y.$$

→ function of y

$$\int_1^2\int_0^3 x^2y\,dxdy = \int_1^2 9y\,dy = 9\cdot\frac{y^2}{2}\Big|_1^2 = \frac{27}{2}.$$

$9\cdot\frac{x^2}{2}\Big|_1^2$

double integral ⟷ iterated integral

4. Fubini's Theorem

If f is continuous on the rectangle,

$R=[a,b]\times[c,d]$, then,

$$\iint_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy.$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

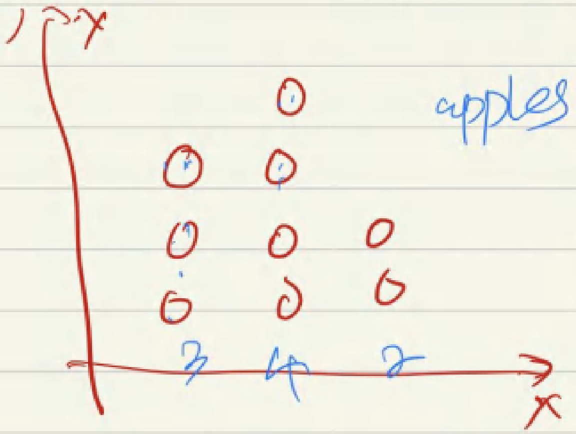
double integral ⟷ iterated integral.

"integrate step by step"!

integration w.r.t 2 variables ⟶ integration w.r.t one variable, twice.

discrete function

idea: Suppose we want to count the apples;



1) count one by one : 9.
double integral.

2) count apples in each column; 3, 4, 2, → $\int f\,dy$
then take the sum 3+4+2 = 9. → $(\int f\,dy)\,dx$

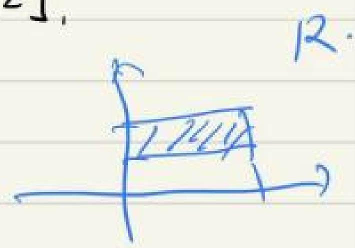
iterated integral

"counting measure" ⟷ Lesbesgue measure □

$$\underbrace{\iint_R (f+g)\,dA = \iint_R f\,dA + \iint_R g\,dA.}_{\uparrow\ \Sigma\ \text{Riemann}}$$

7,

Example $\iint_R (x-3y^2)\,dA$. $R = [0,2]\times[1,2]$.



Solution. 1.

$$\iint_R (x-3y^2)\,dA = \int_0^2 \underbrace{\int_1^2 (x-3y^2)\,dy}_{}\,dx$$

of x

$$\int_1^2 (x-3y^2)\,dy = xy - y^3\Big|_1^2 = x-7, \quad x$$

$$\iint_R (x-3y^2)\,dA = \int_0^2 (x-7)\,dx = \frac{x^2}{2} - 7x\Big|_0^2 = -12$$

Solution 2.

$$\iint_R (x-3y^2)\,dA = \int_1^2\int_0^2 (x-3y^2)\,dx\,dy$$

$$\int_0^2 (x-3y^2)\,dx = \frac{x^2}{2} - 3y^2x\Big|_0^2 = 2-6y^2$$

$$\iint_R (x-3y^2)\,dA = \int_1^2 (2-6y^2)\,dy = 2y-2y^3\Big|_1^2 = -12.$$

Solution 3.

$$\iint_R (x-3y^2)\,dA = \iint_R x\,dA - \iint_R 3y^2\,dA = I_1 - I_2$$

① "constant w.r.t y"

$$I_1 = \int_1^2\int_0^2 x\,dx\,dy = \int_1^2 \boxed{\left(\frac{x^2}{2}\Big|_0^2\right)}\,dy = \frac{x^2}{2}\Big|_0^2 \cdot \underbrace{\int_1^2 dy}_{1}$$

$$= \frac{x^2}{2}\Big|_0^2 = 2, \quad \leftarrow \text{independent of } x$$

② change orders to simplify computation.

$$I_2 = \int_0^2\int_1^2 3y^2\,dy\,dx = 2\cdot\int_1^2 3y^2\,dy = 2\cdot y^3\Big|_1^2$$

$$= 14. \qquad \int_1^2 3y^2\,dy \cdot \underbrace{\int_0^2 dx}_{2}$$

$$\iint_R (x-3y^2)\,dA = 2-14 = -12,$$

$\int \underline{y} \sin \underline{xy}\, dy \longrightarrow$ integration by parts

Example $\iint_R y\sin(xy)\,dA$. $R=[1,2]\times[0,\pi]$

Solution: $\iint_R y\sin(xy)\,dA = \int_0^{\pi}\int_1^2 y\sin(xy)\,\underline{dx}\,dy$

$$\int_1^2 y\sin(xy)\,dx = -\cos(xy)\Big|_{x=1}^{x=2} = \underline{-\cos 2y+\cos y}.$$

$y\,dx = d(xy)$ y is a constant

$$\iint_R y\sin(xy)\,dA = \int_0^{\pi}(-\cos 2y+\cos y)\,dy$$

$$= \frac{-1}{2}\sin 2y + \sin y\Big|_0^{\pi} = 0.$$

$\square$

If $f(x,y) = g(x)\cdot h(y)$ $R=[a,b]\times[c,d]$.

$$\iint_R f(x,y)\,dA = \int_c^d\int_a^b g(x)h(y)\,dx\,dy$$

$$= \int_c^d \Big[\underline{\int_a^b g(x)h(y)\,dx}\Big]\,dy$$

$\rightarrow$ fixed y

a constant independent of y

☆ $$= \int_c^d \underline{h(y)}\Big[\underline{\int_a^b g(x)\,dx}\Big]\,dy = \Big(\underline{\int_a^b g(x)\,dx}\Big)\int_c^d h(y)\,dy. \quad \square$$

Example $\iint_R \sin x\cos y\,dA$ $R=[0,\pi/2]\times[0,\pi/2]$.

Solution $$\iint_R \sin x\cos y\,dA = \int_0^{\pi/2}\sin x\,dx\cdot\int_0^{\pi/2}\cos y\,dy$$

$$= 1\times 1 = 1.$$

$\square$

$$\int_0^{\pi/2}\sin x\,dx = -\cos x\Big|_0^{\frac{\pi}{2}} = 1.$$

symmetry → simplify computations. 9,

Example $\iint_R \frac{xy}{1+x^4}\,dA$ $\qquad R=[-1,1]\times[0,1].$

Solution: $\iint_R \frac{xy}{1+x^4}\,dA = \int_0^1\int_{-1}^1 \frac{xy}{1+x^4}\,dx\,dy.$

$$\int_{-1}^1 \frac{xy}{1+x^4}\,dx = \int_{-1}^0 \frac{xy}{1+x^4}\,dx + \int_0^1 \frac{xy}{1+x^4}\,dx.$$

$$= I_1 + I_2.$$

$$I_1 \overset{x=-t}{=} \int_1^0 \frac{-ty}{1+t^4}\,d(-t) = -\int_0^1 \frac{ty\,dt}{1+t^4} = -I_2$$

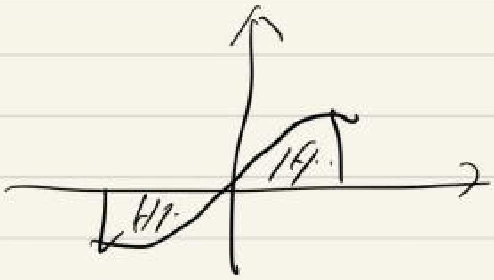
$$\Rightarrow \int_{-1}^1 \frac{xy}{1+x^4}\,dx = 0.$$

$$\Rightarrow \iint_R \frac{xy}{1+x^4}\,dA = 0 = \int_0^1 0\cdot dy$$

□

$\int_{-1}^1 f(x)\,dx = 0$ if $f(x)$ is odd.

$f(-x) = -f(x)$



Double integral $\Leftrightarrow$ iterated integral.

2 variable

one variable 2 1.

Lecture 14.

Double integrals over general regions. (compact).

1. For single integrals, the region over which we integrate is always an interval.

An open set in $\mathbb{R}^1$ is a disjoint union of open intervals.
A closed set in $\mathbb{R}^1$ is a disjoint union of closed intervals.
A compact set in $\mathbb{R}^1$ is a finite disjoint union of bounded closed intervals.

"topologically good"

However, subsets in $\mathbb{R}^2$ are much more complicated than those in $\mathbb{R}^1$. We are mainly interested in compact subsets, that is, bounded closed subsets, D.

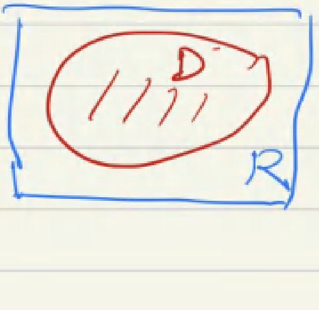
Functions over non-compact sets may have "bad" properties, (unbounded, etc).

Of course, there are compact subsets that are not rectangles. Example: disks.

NOT necessarily compact

Now let D be bounded set. Then D can be enclosed in a rectangle R. Let f be a function over D,
We define a function F over R.

$$F(x,y)=\begin{cases} f(x,y) & \text{if } (x,y)\in D, \\ 0 & (x,y)\notin D. \end{cases}$$



"zero extension".

$f(x,y)$ / D.   domain

$\to F(x,y)$ R Rectagle

2,

. If F is integrable over R, then we define the double integral of f over D by

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA$$

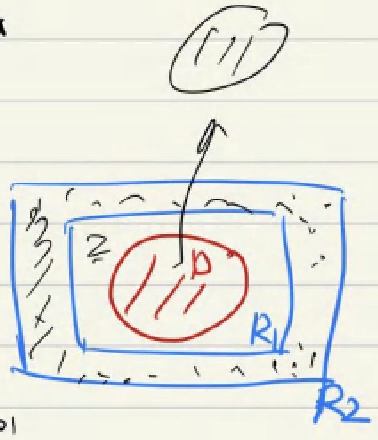
If $f(x,y) \geqslant 0$, $\iint_D f(x,y)\,dA$ represents the volume of the solid between D and the graph of $f(x,y)$.

Remark: The double integral is well-defined (independent of the choices of rectangles R).

We may assume $D \subset R_1 \subset R_2$. Let F_1, F_2 be zero extensions of f over R_1, R_2..

then $\iint_{R_2} F_2(x,y)\,dA - \iint_{R_1} F_1(x,y)\,dA$

$= \iint_{R_2 \setminus R_1} 0\,dA = 0$



Remark when is F integrable?

$f|_D \to$ zero extension

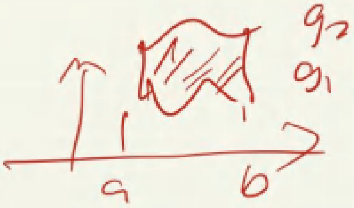
- f is good;
- D is good;

(Example: if $D = ([0,1]\times[0,1]) \cap \mathbb{Q}^2$, $f = 1$. F is everywhere discontinuous.) Not integrable

We are mainly interested in the case where f is continuous

and D is of type I or II, and their unions.

"good domains!"



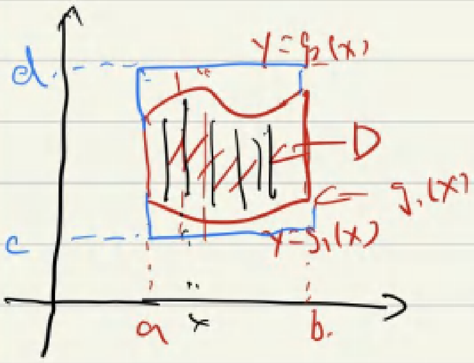
3,

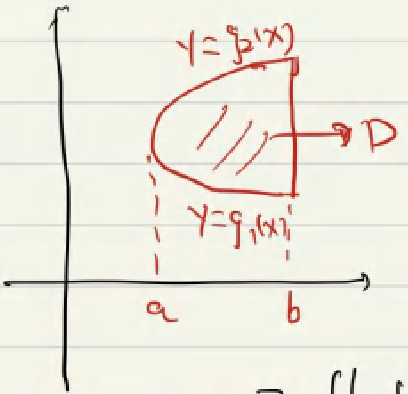
2. A plane region ✓ is said to be of type I. if it lies between the graphs of two continuous functions of x:

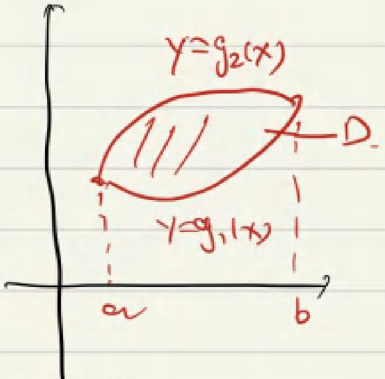
$$D=\{(x,y)\mid a\le x\le b,\ g_1(x)\le y\le g_2(x)\}.$$

where g_1 and g_2 are continuous on $[a,b]$.

type I ✓







$$f(x,y)\ /D \longrightarrow \iint_D f(x,y)\,dA$$

choose a rectangle $R=[a,b]\times[c,d]$ that contains D. ← f,g continuous /[a,b] bounded

Let F be the zero extension.

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA = \int_a^b\int_c^d F(x,y)\,dy\,dx$$

definition ↗ ← Fubini. x

zero extension

$$\text{Fix, } x.\quad \int_c^d F(x,y)\,dy \overset{\text{zero extension}}{=} \int_{g_1(x)}^{g_2(x)} F(x,y)\,dy = \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy.$$

If f is continuous on a type I region D. such that.

$$D=\{(x,y)\mid a\le x\le b,\ g_1(x)\le y\le g_2(x)\}.$$

then

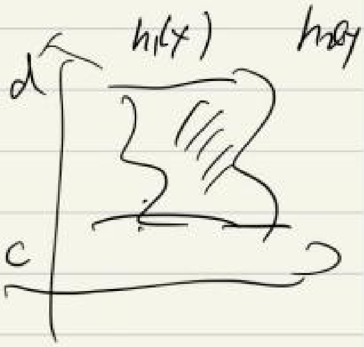
$$\iint_D f(x,y)\,dA = \int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

← g1, g2 constant

$\int_a^b\int_c^d f(x,y)\,dy\,dx$ ← rectangle. iterated integral.

Remark: The inner integral is a function of x, then integrate with respect to x over an interval.

We also define plane regions of type II:

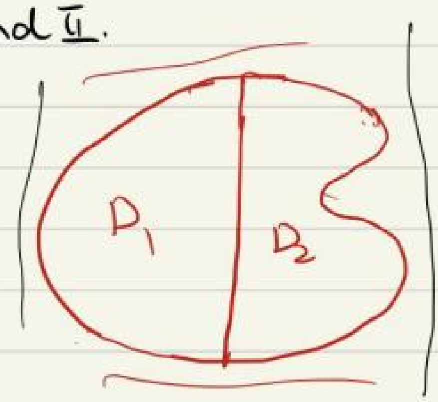


$$D=\{(x,y)\mid c\le y\le d.\ \ h_1(y)\le x\le h_2(y)\}.$$

h_1 and h_2 are continuous

$$\iint_D f(x,y)\,dA=\int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Union of regions of type I and II.



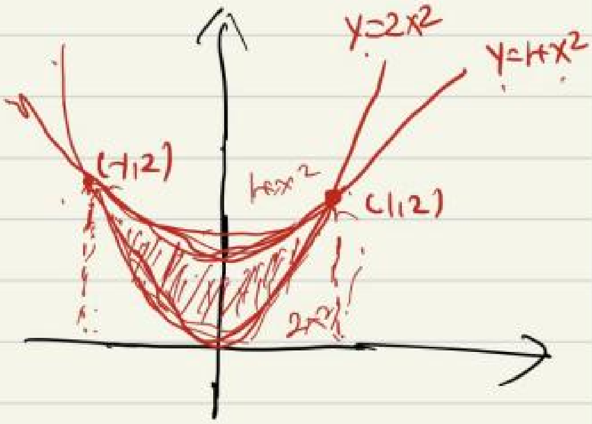
D is NOT of type I or II, but $D=D_1\cup D_2$, D_1 is of type I, D_2 is of type II,

Example. $\iint_D (x+2y)\,dA$. ☆

D: type I, II?

$g_1\ g_2$ / h_1 h_2

D is the region bounded by the parabolas $y=2x^2$, and $y=1+x^2$



Solution. first step:

$D=\{(x,y)\mid -1\le x\le 1, 2x^2\le y\le x^2+1\}$,

type I.

represent D as region of type I / II. find g, h:

x

$$\iint_D (x+2y)\,dA = \int_{-1}^{1}\int_{2x^2}^{1+x^2} (x+2y)\,dy\,dx.$$

$$\int_{2x^2}^{1+x^2} (x+2y)\,dy = xy+y^2\Big|_{2x^2}^{1+x^2}$$

x is constant

fixed x

$$\int_0^{f(x)} g(x)\,dx$$

$$= x(1+x^2) + (1+x^2)^2 - x\cdot 2x^2 - (2x^2)^2$$

$$= -3x^4 - x^3 + 2x^2 + x + 1.$$

$$\iint_D (x+2y)\,dA = \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\,dx$$

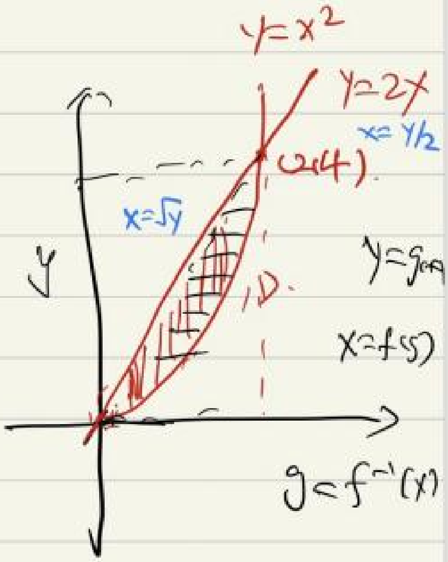
primitive

$$= -\frac{3}{5}x^5 - \frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2} + x\Big|_{-1}^{1} = \frac{32}{15}. \quad \square$$

Example $\iint_D (x^2+y^2)\,dA$

D is the region bounded by the line $y=2x$, and the parabola $y=x^2$.

Solution 1 $D = \{(x,y), \mid 0 \le x \le 2,\ x^2 \le y \le 2x\}.$



$$\iint_D (x^2+y^2)\,dA = \int_0^2 \int_{x^2}^{2x} (x^2+y^2)\,dy\,dx$$

$$\int_{x^2}^{2x} (x^2+y^2)\,dy$$

x is a constant

$$= x^2 y + \frac{y^3}{3}\Big|_{x^2}^{2x}$$

$$= x^2\cdot(2x) + \frac{1}{3}(2x)^3 - x^2\cdot x^2 - \frac{1}{3}(x^2)^3$$

$$= -\frac{x^6}{3} - x^4 + \frac{14}{3}x^3.$$

polynomial of x

6.

$$\iint_D (x^2+y^2)dA$$

$$=\int_0^2 \left(-\frac{x^6}{3}-x^4+\frac{14}{3}x^3\right)dx$$

$$=-\frac{x^7}{21}-\frac{x^5}{5}+\frac{7x^4}{6}\Big|_0^2 = \frac{216}{35}.$$

Solution 2:

$$D=\{(x,y)\mid 0\le y\le 4,\ \tfrac{1}{2}y\le x\le \sqrt{y}\},$$

$$\iint_D (x^2+y^2)dA = \int_0^4 \underline{\int_{y/2}^{\sqrt{y}}} (x^2+\underline{y}^2)\,dx\,dy$$

$$\int_{y/2}^{\sqrt{y}} (x^2+y^2)dx = \frac{x^3}{3}+y^2x\Big|_{y/2}^{\sqrt{y}}$$

$$=\frac{y^{3/2}}{3}+y^{5/2}-\frac{y^3}{24}-\frac{y^3}{2}. \longrightarrow y$$

$$\iint_D (x^2+y^2)dA = \int_0^4 \left(\frac{y^{3/2}}{3}+y^{5/2}-\frac{y^3}{24}-\frac{y^3}{2}\right)dy$$

$$=\frac{2}{15}y^{5/2}+\frac{2}{7}y^{7/2}-\frac{13}{96}y^4\Big|_0^4 = \frac{216}{35}.. \quad \square$$

$$\iint_R f(x,y)dA = \int_a^b\int_c^d f\,dy\,dx$$

$$=\int_c^d\int_a^b f\,dx\,dy.$$

the order

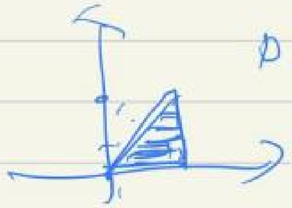
☆ useful technique ☆

3. Changing the order of integration.

It is ~~often~~ useful to change the order of integration.

(when D is both of type I and type II).

Example. $\underset{\sim\sim}{\int_0^1 \int_y^1} \frac{\sin x}{x}\, dx\, dy \longrightarrow \underline{\text{type II}}$



Solution : $\int \frac{\sin x}{x} dx$ is NOT a fundamental function.

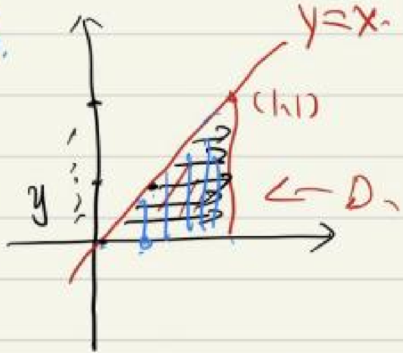
$\{(x,y) \mid 0 \le y \le 1,\ y \le x \le 1\}$

$\|$

$\{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le x\}$

$$\int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy$$

$$= \iint_{D} \frac{\sin x}{x}\, dA,$$



y, x

↓

type I

$$\iint_D \frac{\sin x}{x} dA \quad \int_0^1 \int_0^x \frac{\sin x}{x}\, dy\, dx.$$

But $\int_0^x \frac{\sin x}{x} dy = \frac{\sin x}{x} \cdot x = \sin x.$

$$\Rightarrow \iint_D \frac{\sin x}{x} dA = \int_0^1 \sin x\, dx = -\cos x \Big|_0^1 = 1 - \cos 1.$$

□

$\iint_{D} f(x,y)\,dA$ $\qquad \int_0^x dy = x$

if D as type I and type II.

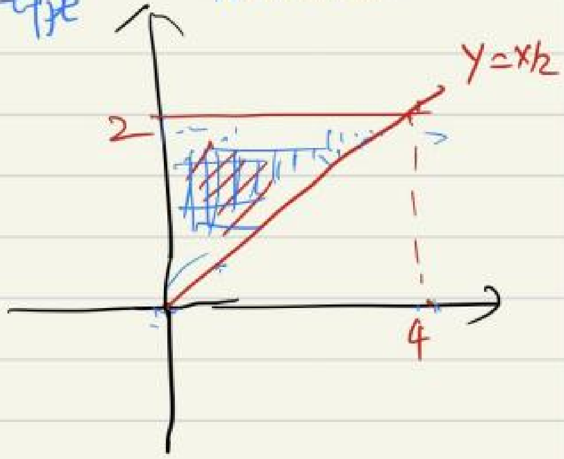
$$\iint_D f(x,y)\,dA = \boxed{\int_a^b \int_{g_1}^{g_2} f(x,y)\,dy\,dx} = \boxed{\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy}.$$

a family of intervals

type: vertical

type horizontal

Example: $\boxed{\int_0^4\int_{x/2}^2} e^{y^2}\,dy\,dx. \xrightarrow{} \text{type I}.$



$D = \{(x,y) \mid 0 \le x \le 4,\ x/2 \le y \le 2\}$

$= \{(x,y) \mid 0 \le y \le 2,\ 0 \le x \le 2y\}.$

$$\boxed{\int_0^4\int_{x/2}^2} e^{y^2}\,dy\,dx = \iint_D e^{y^2}\,dA = \boxed{\int_0^2\int_0^{2y}} e^{y^2}\,dx\,dy.$$

$$\int_0^{2y} e^{y^2}\,dx = 2ye^{y^2}. \longrightarrow 2ye^{y^2} = \frac{d}{dy}e^{y^2}$$

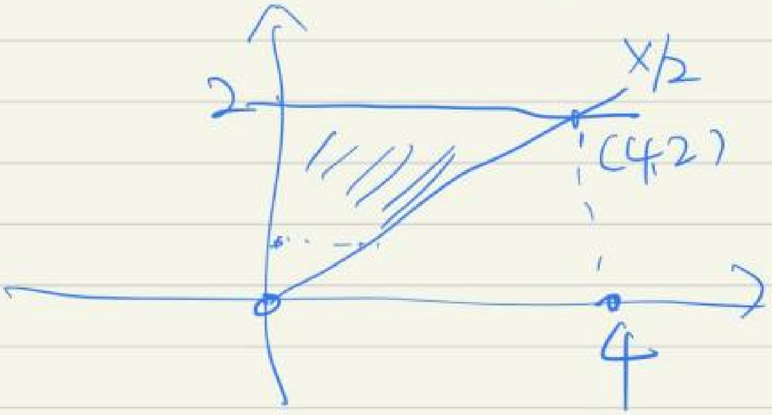
chain rule.

$$\Rightarrow \int_0^4\int_{x/2}^2 e^{y^2}\,dy\,dx = \int_0^2 2ye^{y^2}\,dy$$

$$= e^{y^2}\Big|_0^2 = e^4 - 1.$$

type I

D.



$y = x/2$

$x = 2y$

type II

$\boxed{= \{0 \le y \le 2 \mid 0 \le x \le 2y)}$

$\iint f \longleftrightarrow$ volume

natural.

4. Properties of double integrals.

(1) $$\iint_D [f(x,y)+g(x,y)]\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA.$$

(2) $$\iint_D c f(x,y)\,dA = c\iint_D f(x,y)\,dA,$$ c . constant

(3). if $f(x,y) \geqslant g(x,y)$. $\forall (x,y)\in D$.

$$\iint_D f(x,y)\,dA \geqslant \iint_D g(x,y)\,dA$$

(4) If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries, then

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA.$$

(5). $$\iint_D 1\,dA = \text{Area}(D.)$$

Example. Estimate the integral $\iint_D e^{\sin x\cos y}\,dA$.

D : disk with center the origin and radius 2.

Solution. $-1 \leq \sin x \leq 1$, $-1, \leq \cos y \leq 1$.

$\Rightarrow$ $-1 \leq \sin x\cos y \leq 1$.

$e^{-1} \leq e^{\sin x\cos y} \leq e$. $A(D) = 4\pi$

$$\Rightarrow \frac{4\pi}{e} \leq \iint_D e^{\sin x\cos y}\,dA \leq 4\pi e$$

✓

(1) $f(-x,y)=-f(x,y)$

(2) $f(x,-y)=-f(x,y)$

1.

over "symmetric" domains

Lecture 15.

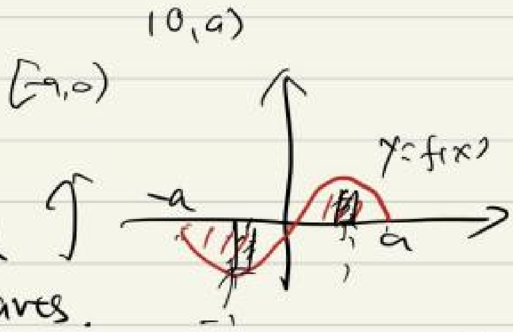
1. An interesting example.

Sometimes symmetries make computations simpler.

Fact If $f(x)$ is odd ($f(-x)=-f(x)$), then

$$\int_{-a}^{a} f(x)dx=0.$$

Geometrically,



The signed area between $[-a,a]$ and the graph $y=f(x)$. have 2 parts.

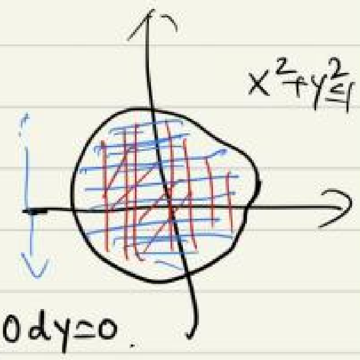
The signed area of these parts are opposite to each other.

$$\int_{-a}^{0} f(x)dx \overset{x=-t}{=} \int_{a}^{0} f(-t)\,d(-t) = \int_{a}^{0} -f(t)\,dt = -\int_{0}^{a} f(t)\,dt \quad \square$$

odd functs

Example $\iint_D (x+2y)\,dA$. $D=\{(x,y) \mid x^2+y^2\leq 1\}$.

Solution. $\iint_D (x+2y)dA = \iint_D x\,dA + \iint_D 2y\,dA$.



$$\iint_D x\,dA = \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x\,dx\,dy$$

but $\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x\,dx = 0 \Rightarrow \iint_D x\,dA = \int_{-1}^{1} 0\,dy = 0$.

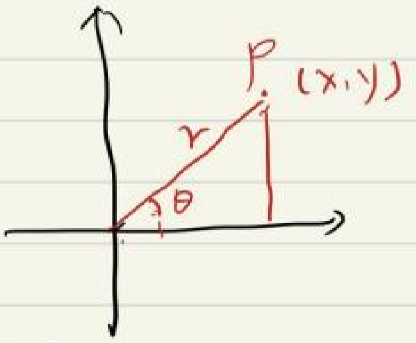
Similarly, $\iint_D 2y\,dA = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2y\,dy\,dx = 0$

$\Rightarrow \iint_D (x+2y)\,dA = 0$ *

difficulty $\begin{cases} (1) \text{ function} \\ (2) \text{ domain/region} \cdots\cdots ☆ \end{cases}$

2,

2, polar coordinates:

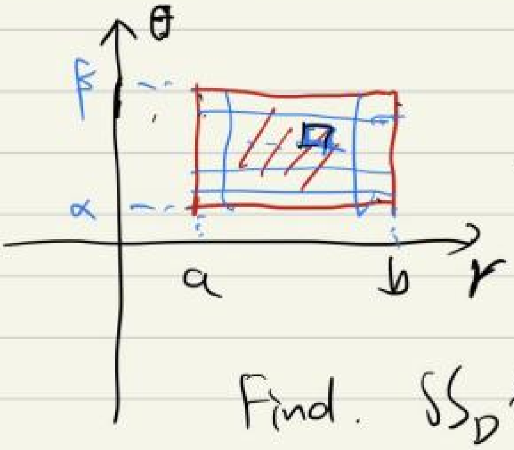


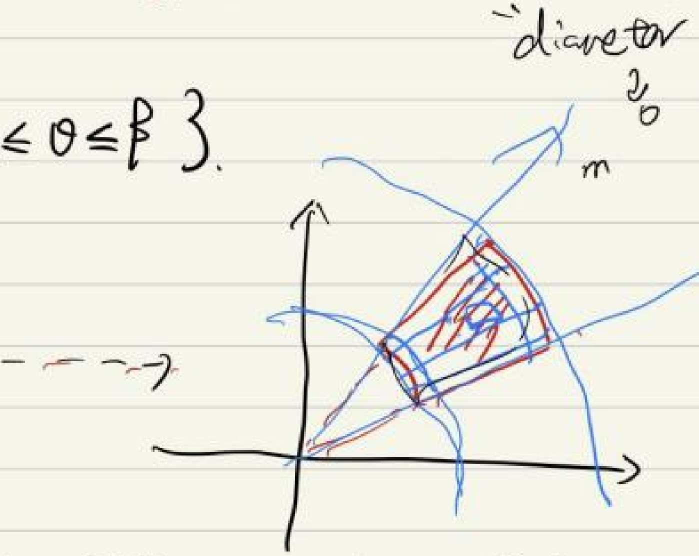
$\begin{cases} x = r\cos\theta \\ y = r\sin\theta. \end{cases}$ $\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arctan\frac{y}{x}. \end{cases}$

(general x,y)
Not 1:1,
$(0,\theta) \longrightarrow O.$

polar rectangle

$$R = \{(r,\theta) \mid a \le r \le b,\ \alpha \le \theta \le \beta\}.$$



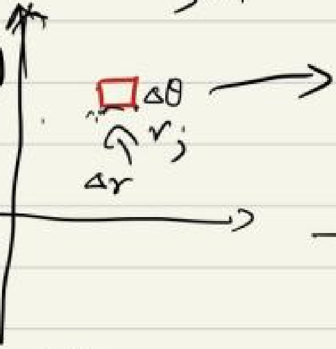


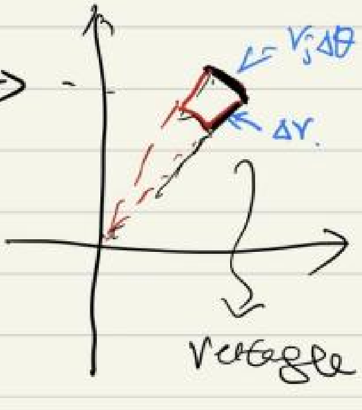
Find. $\iint_D f(x,y)\,dA$, D polar rectangle.

divide $[a,b]$ into m subintervals $[r_{i-1}, r_i]$ of equal width $\Delta r = (b-a)/m$,
$[\alpha,\beta]$ ⋯ n $[\theta_{j-1}, \theta_j]$ ⋯ $\Delta\theta = (\beta-\alpha)/n$.

$\Rightarrow$ mn, sub-rectangles ⋯ $\Delta r \Delta\theta \approx$ small

Let $R_{ij} = \{(r,\theta) \mid r_{i-1} \le r \le r_i,\ \theta_{j-1} \le \theta \le \theta_j\}$
when m, n are sufficiently large,
the image of rectangle in xy-plane
is a rectangle, with area
$\underbrace{(r_j\Delta\theta)\cdot\Delta r}$. + higher order terms





We define the Riemann sum.

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i\cos\theta_j,\ r_i\sin\theta_j)\Delta A_i \longrightarrow$$ area of polar rectangles

$$=\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i\cos\theta_j,\ r_i\sin\theta_j)\, r_i\,\Delta r\Delta\theta.$$

when $m,n\to\infty$. we get.

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta\int_a^b f(r\cos\theta,\ r\sin\theta)\,\underline{\underline{r}}\,dr\,d\theta.$$

Remark In this argument, we extend the definition of double integrals. "small rectangles" need not be the same. We only need to assume that the diameter of subdivisions $\to 0$.

↖ maximum of diameters of all rectangle

Change to polar coordinates in a double integral:

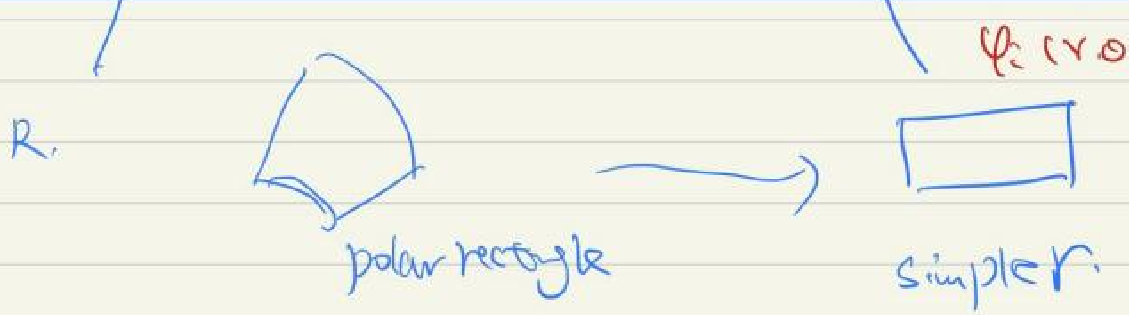
If f is continuous on a polar rectangle R given by

$$0\le a\le r\le b\ ,\quad \alpha\le\theta\le\beta\ ,\ 0\le\beta-\alpha\le 2\pi.$$

then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)\,\underline{\underline{r}}\,dr\,d\theta.$$

$=\|\det\varphi\|$

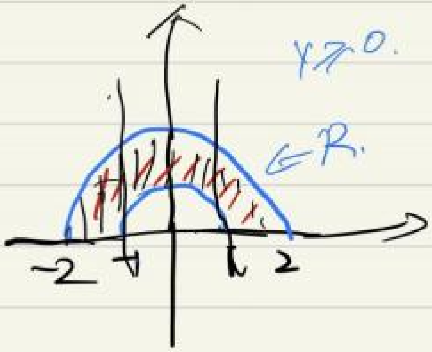


$\varphi: (r,\theta)\to(x,y) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$

Example. $\iint_R (3x + 4y^2)\,dA$.

R: the region in the upper half-plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$

Solution.



←

$$R = \{(x,y) \mid y \geq 0,\ 1 \leq x^2+y^2 \leq 4\}$$

$$= \{(r,\theta) \mid 1 \leq r \leq 2,\ 0 \leq \theta \leq \pi\}$$

⇐ rectangle.

simplify the domain.

$$\iint_R (3x+4y^2)\,dA$$

$$= \int_0^{\pi}\int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\,\underline{\underline{\textcircled{r}}}\,\underset{\sim}{d}r\,d\theta$$

← constants

$$\int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\,r\,dr = \int_1^2 3r^2\cos\theta + 4r^3\sin^2\theta\,dr$$

$$= r^3\cos\theta + r^4\sin^2\theta\Big|_{r=1}^{2} = \underline{7\cos\theta + 15\sin^2\theta}.$$

$$\iint_R (3x^2+4y^2)\,dA = \int_0^{\pi} (7\cos\theta + 15\sin^2\theta)\,d\theta$$

$$= \int_0^{\pi}\left[7\cos\theta + \frac{15}{2}(1-\cos 2\theta)\right]d\theta$$

$$= 7\sin\theta + \frac{15}{2}\theta - \frac{15}{4}\sin 2\theta\Big|_0^{\pi} = \underset{\sim}{\frac{15\pi}{2}}.$$

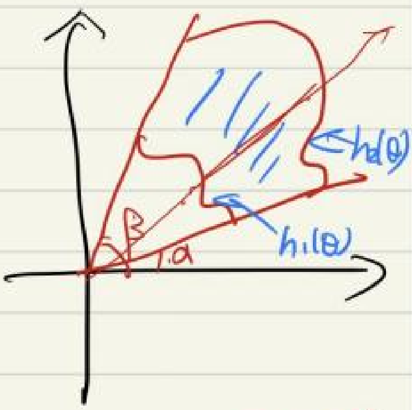
"type Ⅱ": (r,θ)

If f is continuous on a polar region of the form

$$D=\{(r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$$

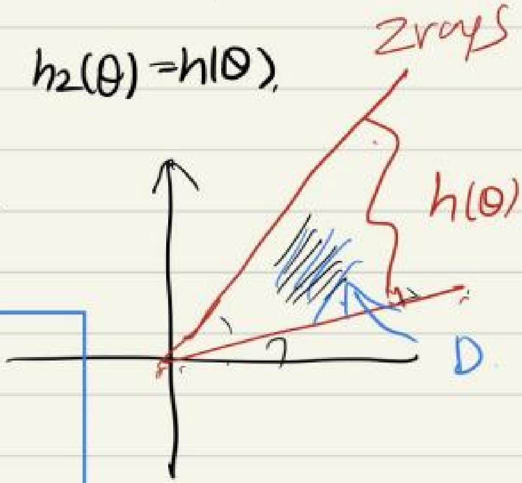
then.

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$



In particular. $f(x,y)=1$, $h_1(\theta)=0$. $h_2(\theta)=h(\theta)$.

D: the region bounded by $\theta=\alpha$, $\theta=\beta$, $r=h(\theta)$:



$$A(D) = \iint_D 1\,dA = \int_\alpha^\beta \int_0^{h(\theta)} r\,dr\,d\theta$$

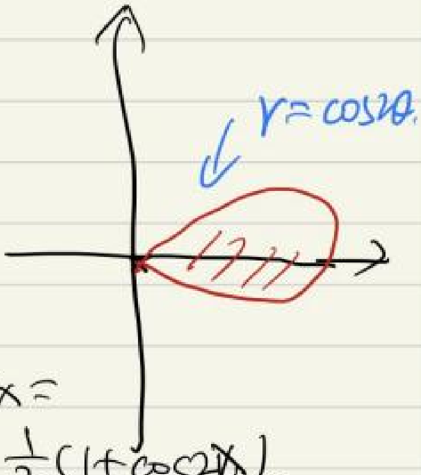
$$= \int_\alpha^\beta \left(\frac{r^2}{2}\right)\Big|_0^{h(\theta)} d\theta = \frac{1}{2}\int_\alpha^\beta h(\theta)^2\,d\theta.$$

Example. $r=\cos 2\theta$. $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$.

Solution. $A(D) = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta\,d\theta$

$= \frac{1}{4}\int_{-\pi/4}^{\pi/4} (1+\cos 4\theta)\,d\theta$

$= \frac{1}{4}\left(\theta + \frac{1}{4}\sin 4\theta\right)\Big|_{-\pi/4}^{\pi/4} = \frac{\pi}{8}$



$\cos^2 x = \frac{1}{2}(1+\cos 2x)$.

Use technique from double integral $\longrightarrow \underline{\int e^{-x^2} dx} \longrightarrow$ to compute. integral of one variables.

6.

3. An interesting integral:

Gaussian integral

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}}$$

statistics

normal distribution

Proof: Step 1:

double integral over unbounded domains

Let D_a be the disk $x^2+y^2 \leq a^2$.

$$\iint_{D_a} e^{-x^2-y^2} dA$$ polar coordinate

$$= \int_0^{2\pi} \int_0^a r e^{-r^2} dr\, d\theta.$$

$D = \{(r,\theta) \mid 0 \leq r \leq a,\ 0 \leq \theta \leq 2\pi\}$

$$\int_0^a \boxed{r} e^{-r^2} dr = -\frac{1}{2} e^{-r^2}\Big|_0^a = \frac{-1}{2}\left(e^{-a^2}-1\right).$$

☆ →

$$\iint_{D_a} e^{-x^2-y^2} dA = -\int_0^{2\pi} \frac{1}{2}\left(e^{-a^2}-1\right) = \pi\left(1-e^{-a^2}\right).$$

$$\frac{d}{dr} e^{-r^2} = -2r \cdot e^{-r^2}$$

Step 2: Let $a \to \infty$.

$$\iint_{\mathbb{R}^2} e^{-x^2-y^2} dA = \pi. \quad \Longleftarrow \quad \lim D_a = \mathbb{R}^2$$

☆ Step 3.

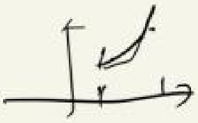
$\mathbb{R}^2 = \lim D_a$. $\quad \shortparallel \quad (-\infty,\infty) \times (-\infty,\infty)$

$$\iint_{\mathbb{R}^2} e^{-x^2-y^2} dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2} \cdot e^{-y^2} dx\, dy$$

$$= \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2} dy\right) = \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2$$

□

$$\int_a^b \int_c^d g(x)h(y)\, dy\, dx = \left(\int_a^b g(x) dx\right) \cdot \left(\int_c^d h(y) dy\right)$$



Lecture 16.

$$\begin{cases} \text{length} & y=f(x) \quad \int_a^b \sqrt{1+[f'(x)]^2}\,dx \\ \text{area} & z=f(x,y) \quad \iint_D \sqrt{1+(f_x(x,y))^2+(f_y(x,y))^2}\,dA \end{cases}$$

☆☆

1. Surface Area

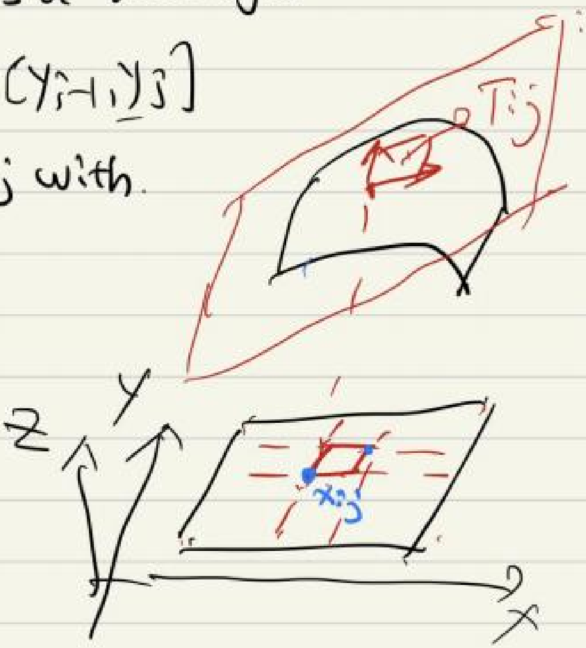
Let S be a surface with equation $z=f(x,y)$, where f has continuous partial derivatives. (C^1).

We first assume $f(x,y)\geqslant 0$, D is a rectangle

$[x_{i-1},x_i]\times[y_{j-1},y_j]$

Divide D into small rectangles, R_{ij} with area $\Delta A=\Delta x\Delta y$.

Let $P_{ij}=(x_i,y_j)\in R_{ij}$



The tangent plane to S at P is an approximation to S near P_{ij}.

Let ΔT_{ij} be the area of this tangent plane that lies directly above R_{ij}.

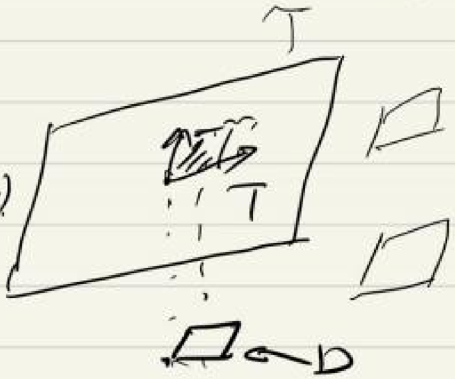
$$A(S)=\lim_{m,n\to\infty}\sum\sum \Delta T_{ij}$$

$$\frac{\Delta T_{ij}}{\Delta A}\approx\sqrt{1+f_x^2+f_y^2}$$

If $\Delta T_{ij}=g(x_i,y_j)\Delta A$

$$\Rightarrow A(S)=\lim_{m,n}\sum\sum g(x_i,y_j)\Delta A=\iint_D g(x,y)\,dA.$$

What is $g(x_i\,y_j)$? (pure linear algebra).



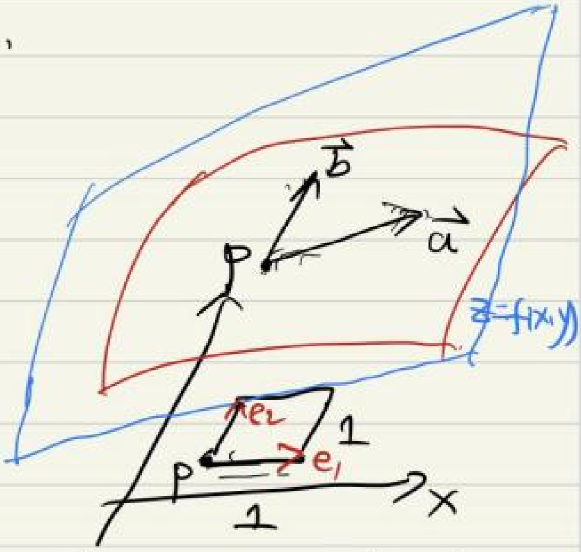
We have a plane, in space,

Let D be a rectangle in $\mathbb{R}^2$, (x-y-plane)

T be the parallelogram lying above D,

The A(T)/A(D) is a constant, independent of the choices. of D.

Now we take a $\overset{D.}{\text{square}}$ of length 1.
whose lower left vertex is (x, y),



then the vectors,

$$\vec a = (1, 0\ \ f_x(x,y))$$

$$\vec b = (0, 1\ ,\ f_y(x,y)).$$

- $\vec a, \vec b$ lies in the tangent plane. ⟸ by definition
- the projections of $\vec a, \vec b$ are e_1, e_2. ⟸ ignore the z component
- The parallelgram $\overset{T}{\text{over}}$ the $\underset{D}{\text{square}}$ is exactly the parallelgram spanned by $\vec a, \vec b$,

$$\text{Area}(T) = |\vec a \times \vec b|,$$

$$\text{but}\quad \vec a \times \vec b = \begin{vmatrix} \vec i & \vec j & \vec k \\ 1 & 0 & f_x(x,y) \\ 0 & 1 & f_y(x,y) \end{vmatrix} = -f_x(x,y)\vec i - f_y(x,y)\vec j + \vec k$$

$$A(T)/A(D) = \text{Area}(T) = \sqrt{1 + f_x^2(x,y) + f_y^2(x,y)}.$$

In conclusion,

The area of the surface with $z=f(x,y)$ $(x,y)\in D$, where f_x and f_y are continuous. is

$$A(S) = \iint_D \sqrt{[f_x(x,y)]^2+[f_y(x,y)]^2+1}\,dA$$

Remark: It is easy to generalize to the case where D is a general region:

ˇApproximate D with rectangles!?... □

Example $z=x^2+y^2$, D= disk with center the origin and radius 3,

Solution. $\sqrt{1+f_x^2+f_y^2} = \sqrt{1+(2x)^2+(2y)^2}$

$= \sqrt{1+4(\underbrace{x^2+y^2}))}. \longrightarrow r^2$

$\{(r,\theta) \mid 0\le r\le 3,\ 0\le \theta\le 2\pi\}.$

$$A = \iint_D \sqrt{1+4(x^2+y^2)}\,dA$$

↓ polar coordinates $\left\{\begin{array}{l}\text{region}\\ \text{function}\end{array}\right.$

$$= \int_0^{2\pi}\int_0^3 \sqrt{1+4r^2}\,r\,dr\,d\theta$$

$$\int_0^3 \sqrt{1+4r^2}\,r\,dr = \int_0^3 \frac{1}{8}\sqrt{1+4r^2}\,(8r)\,dr = \frac{1}{8}\cdot\frac{2}{3}(1+4r^2)^{3/2}\Big|_0^3$$

$$\Rightarrow A = 2\pi\cdot\frac{1}{8}\cdot\frac{2}{3}(1+4r^2)^{3/2}\Big|_0^3 = \frac{\pi}{6}(37\sqrt{37}-1),$$

interval / rectangle ⟶ rectangular box

4.

2. Triple integral. over rectangular boxes

First assume that f is defined on a rectangular box:

$$B = \{(x,y,z) \mid a \leq x \leq b,\ c \leq y \leq d,\ r \leq z \leq s\}. = [a,b]\times[c,d]\times[r,s]$$

Divide the interval $[a,b]$ into l subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b-a)/l$.

... $[c,d]$ ... m $[y_{j-1}, y_j]$ $\Delta y = (d-c)/m$.

$[r,s]$ n $[z_{k-1}, z_k]$ $\Delta z = (s-r)/n$

Divide the box B into lmn sub-boxes:

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

Each box has volume $\Delta V = \Delta x \Delta y \Delta z$.

Then we form the triple Riemann sum

$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta V$$

↖ sample point in B_{ijk}

Definition The triple integral of f over the box B is.

$$\iiint_B f(x,y,z)\, dV = \lim_{l,m,n\to\infty} \sum f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta V.$$

if this limit exists.

Fubini's theorem for triple integrals

If f is continuous on the rectangular box $B = [a,b] \times [c,d] \times [r,s]$, then

$$\iiint_B f(x,y,z)\, dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\, dx\, dy\, dz$$

order three symoble $\{x, y, z\}$ x, y, z / x, z, y $\longrightarrow$ 6.

5.

Remark: There are six $(= 3!)$ possible orders in which we can integrate, all of which give the same value.

Example $\iiint_B xyz^2\, dV.$ $B = [0,1] \times [-1,2] \times [0,3]$

$xyz^2 \rightarrow \frac{x^2}{2}yz^2$

Solution

$$\iiint_B xyz^2\, dx\,dy\,dz = \int_0^3 \int_{-1}^2 \int_0^1 \underline{x}yz^2\, \underline{dx}\,\underline{dy}\,\underline{dz}$$

$$= \int_0^3 \int_{-1}^2 \frac{x^2yz^2}{2}\Big|_{x=0}^{x=1} dy\,dz = \int_0^3 \int_{-1}^2 \frac{yz^2}{2}\, dy\,dz \leftarrow \text{duble integral}$$

$$= \int_0^3 \frac{y^2z^2}{4}\Big|_{y=-1}^{2} dz = \int_0^3 \frac{3z^2}{4}\, dz = \frac{z^3}{4}\Big|_0^3 = \frac{27}{4}.$$

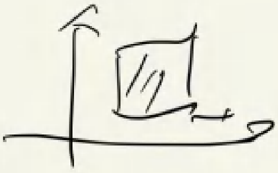
3. Triple integral over a general region E ← bounded

We enclose E in a box B. Then define F to be the zero extension of f to B. By definition.

$$\iiint_E f(x,y,z)\,dV = \iiint_B F(x,y,z)\,dV.$$

The integral exists if f is continuous and the boundary of E is reasonably smooth.

type I in $\mathbb{R}^2$



$\{(x,y))\mid x\in[a,b],$
$u_1(x)\le y\le u_2(x))$

We restrict our attention to continuous functions f and to certain simple types of regions. A solid region E is said to be of type 1 if it lies between the graphs of two continuous functions of x and y, that is,

$$E=\{(x,y,z)\mid (x,y)\in D,\ u_1(x,y)\underline{\le z\le} u_2(x,y)\}.$$

Then.

function of (x,y)

$$\iiint_E f(x,y,z)\,dV=\iint_{\underline{D}}\left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right]dA.$$

In particular, if the projection D of E onto the xy-plane is a type I plane region, then.

$$E=\{(x,y,z)\mid \overset{D}{\underbrace{a\le x\le b,\ g_1(x)\le y\le g_2(x)}},\ u_1(x,y)\le z\le u_2(x,y)\}.$$

$$\boxed{\iiint_E f(x,y,z)\,dV=\int_a^b\underbrace{\int_{g_1(x)}^{g_2(x)}}\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx.}$$

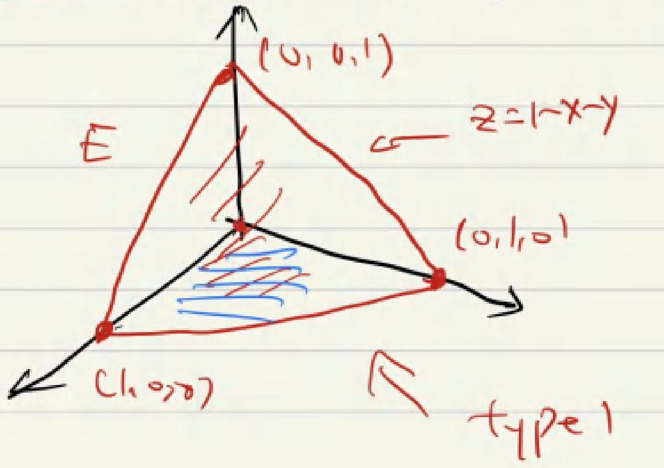
E → type 2/2

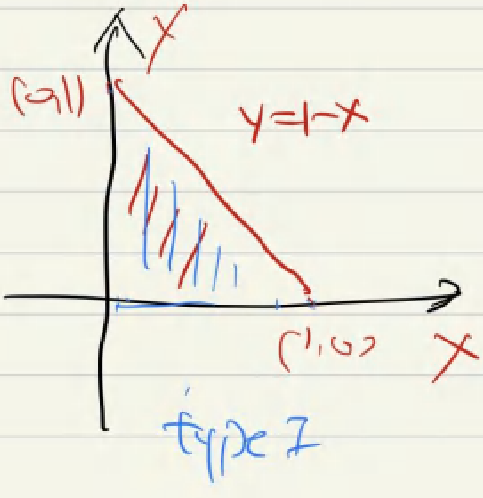
D → type 2/II

Example $\iiint_E z\,dV.$

E is the solid tetrahedron bounded by the four planes $\underline{x=0,\ y=0,\ z=0}$, and $x+y+z=1.$

Solution.





$$E = \{ (x,y,z) \mid 0 \le x \le 1,\ 0 \le \underset{\frown}{y} \le 1-x,\ 0 \le z \le 1-x-y \},$$

← of x, y.

$$\iiint_E z\,dV = \int_0^1 \int_0^{1-x} \underbrace{\int_0^{1-x-y}}\ z\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \frac{z^2}{2}\Big|_{z=0}^{z=1-x-y} dy\,dx$$

$\longrightarrow$ (x, y)

$$= \frac{1}{2}\int_0^1 \int_0^{1-x} (\underset{\smile}{1-x-y})^2\,dy\,dx \longrightarrow$$

~~tri~~ triangle D

$$= \frac{1}{2}\int_0^1 \int_0^{1-x} -\frac{(1-x-y)^3}{3}\Big|_{y=0}^{1-x} dx$$

$$= \frac{1}{6}\int_0^1 \underset{\smile}{(1-x)}^3 dx = \frac{1}{6}\left(-\frac{(1-x)^4}{4}\right)\Big|_0^1 = \frac{1}{24}.$$

"symmetric"

A solid region is of type 2 if it is of the form

$$E = \{ (x,y,z) \mid \underbrace{(y,z) \in D}, \ u_1(y,z) \le x \le u_2(y,z) \}.$$

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx \right] dA.$$

A solid region is of type 3 if it is of the form

$$E = \{ (x,y,z) \mid \underbrace{(x,z) \in D}, \ u_1(x,z) \le y \le u_2(x,z) \}.$$

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,z}^{u_2(x,z).} f(x,y,z)\,dy \right] dA$$

triple integral ← double integral

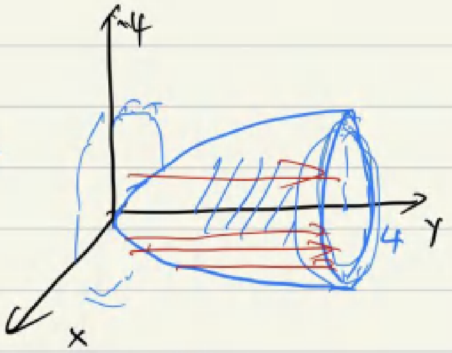
iterated integral

3 2

Example $\iiint_E \sqrt{x^2+z^2}\,dV$.

E: the region bounded by the paraboloid $y=x^2+z^2$ and the plane $y=4$.

Solution type 3.



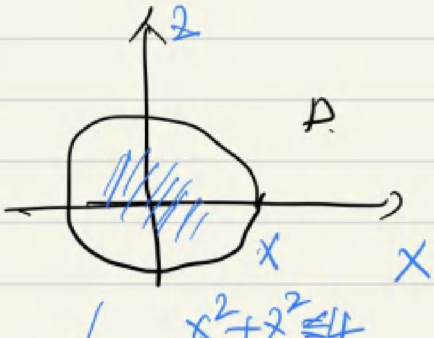
$$\iiint_E \sqrt{x^2+z^2}\,dV$$

polar coordinate r^2

$$= \iint_D \left[\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\,dy\right] dA$$

constant w.r.t. y

$$= \iint_D (4-x^2-z^2)\sqrt{x^2+z^2}\,dA.$$



$x^2+z^2 \leq 4$

$x = r\cos\theta \quad z = r\sin\theta$

$$= \int_0^{2\pi}\int_0^{2} (4-r^2)\,r\,\boxed{r\,dr\,d\theta}$$

polar rectangle

$$= \int_0^{2\pi}\int_0^{2} (4r^2-r^4)\,dr\,d\theta$$

$$= 2\pi\cdot\left(\frac{4r^3}{3}-\frac{r^5}{5}\right)\Big|_0^2 = \frac{128\pi}{15}.$$

$f = g(\theta)\,h(r)$

$4r^2-r^4$

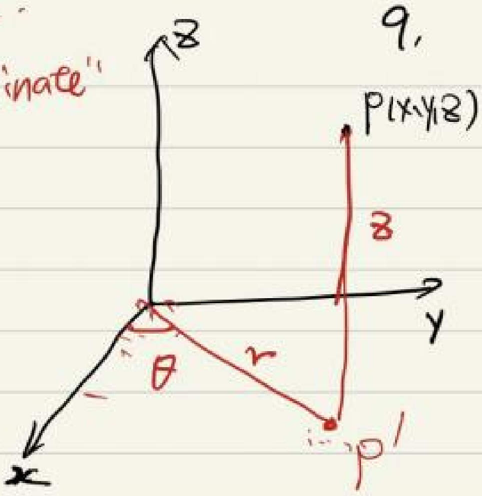
1

$\int_0^{2\pi} 1\,d\theta \cdot \int_0^{2} (4r^2-r^4)\,dr$

4. Cylindrical coordinates.

In cylindrical coordinate system, a point P is three dimensional space is represented by the ordered triple (r, θ, z), where r, and θ are polar coordinates of the projection of P onto the xy-plane and z is the directed distance from the xy-plane to P.

"product" of "polar coordinate in xy-plane" and "in z - usual coordinate"



$$\boxed{x = r\cos\theta,\ y = r\sin\theta \quad z = z.}$$

$$\boxed{r = \sqrt{x^2+y^2},\ \theta = \arctan\frac{y}{x} \quad z = z}$$

Example: cylindrical coordinates $(2, \frac{2}{3}\pi, 1)$.

$\Rightarrow$ rectangular coordinates. $(-1, \sqrt{3}, 1)$.

$x = 2\cdot\cos\frac{2\pi}{3}$

$y = 2\sin\frac{2\pi}{3}$

rectangular coordinates $(3, -3, -7)$.

$\tan\theta = -1$

$\Rightarrow$ cylindrical coordinates.

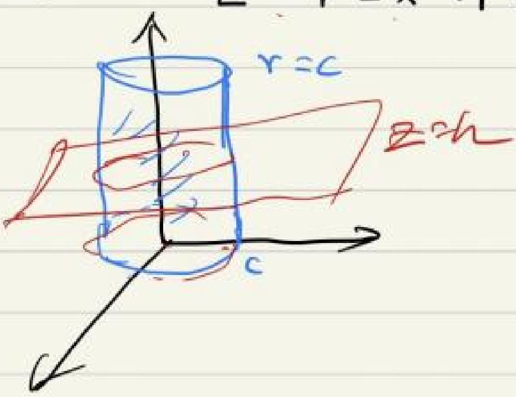
$(\underset{\sim}{3\sqrt{2}}, \frac{7}{4}\pi + 2n\pi, -7)$.

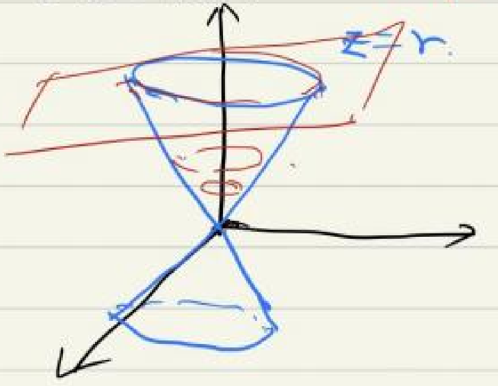
Example $\underline{r = c}$ defines surfaces: $\underline{\text{cylinders}}$,

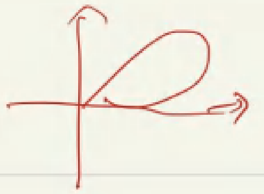
Example: $z = r$.

$z = k \Rightarrow r = k$, a circle. These traces suggest that the equation is a cone.

$\Leftrightarrow$. $z^2 = r^2 = x^2+y^2$, (a cone).







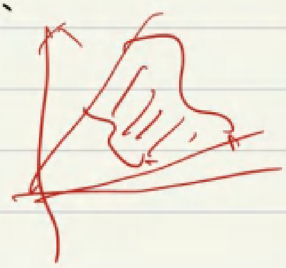
← $\{(x,y) \in D, u_1 \le z \le u_2\}$

Suppose that E is a type I region whose projection D onto the xy-plane is conveniently described in polar coordinates.

In particular, if D is of type II:

$$E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}.$$

$$D = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}.$$



$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA.$$

↖ function of (x,y)

$\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz$ is a function of (x,y)

$$\boxed{\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,r\sin\theta)}^{u_2(r\cos\theta,r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta}$$

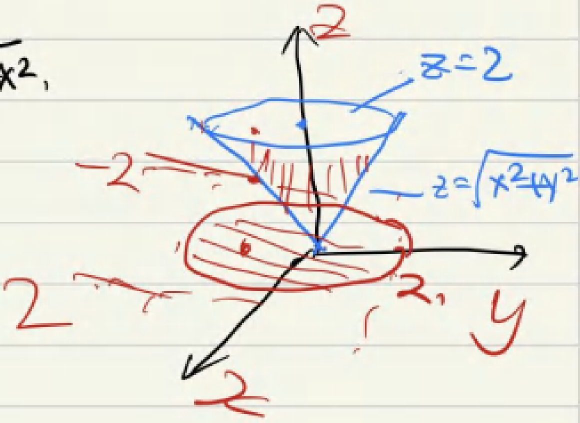
Example: $$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2)\,dz\,dy\,dx$$

$\int f(x,y,z)dz \to (r,\theta)$
$= \int f(r\cos\theta, r\sin\theta, z)\,dz$

Solution

$$E = \{(x,y,z) \mid -2 \le x \le 2,\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2},\ \sqrt{x^2+y^2} \le z \le z \le 2\}$$

$$D = \{(x,y) \mid x^2+y^2 \le 4\}.$$



lower surface: $z = \sqrt{x^2+y^2}$,

upper surface: $z = 2$,

1/.

In cylindrical coordinates.

$$E = \{(r,\theta,z) \mid \underbrace{0 \le \theta \le 2\pi,\ 0 \le r \le 2}_{D},\ \underline{r \le z \le 2}\}.$$

$$x^2+y^2=r^2$$

$$\iiint_E (x^2+y^2)\,dV$$

$$= \int_0^{2\pi}\int_0^2\int_r^2 \underline{r^2}\,\underline{r}\,dz\,dr\,d\theta.$$

$$= \int_0^{2\pi} d\theta \int_0^2 \underline{r^3(2-r)}\,dr.$$

$$= 2\pi\left[\tfrac{1}{2}r^4 - \tfrac{1}{5}r^5\right]_0^2 = \frac{16}{5}\pi$$

□

$r^3\cdot(2-r)$

cylindrical coordinates:

$E \longrightarrow$ type 1.

$D \longrightarrow$ simple to dessribe in polar coordinate

spherical cooridnates; } tomorrow.

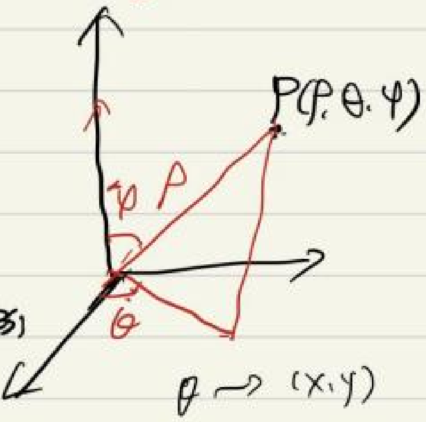
change of variables }

Lecture 17.

change of variables.

1. Spherical coordinates:

simplifies the evaluation of triple integrals over regions bounded by spheres or cones.



The spherical coordinates: (ρ, θ, φ)

$(\rho \geq 0)$

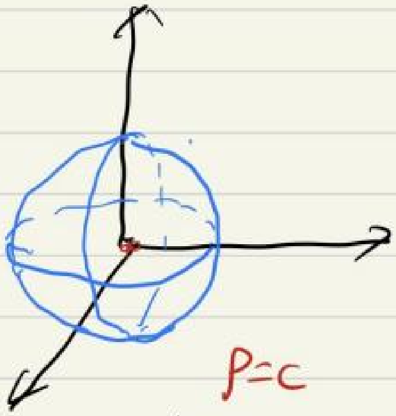
$\rho = |OP|$ is the distance from the origin to P;

θ: same angle as in cylindrical coordinates;

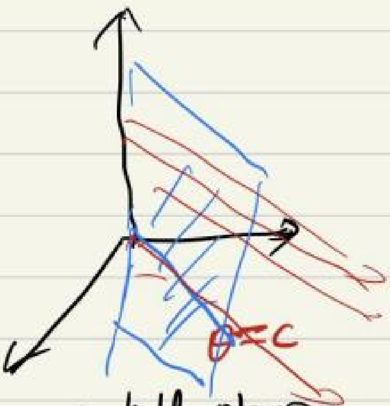
ϕ: the angle between the positive z-axis and the line segment OP. $(0 \leq \phi \leq \pi)$.

The spehorical coordinate system is especially useful in problems where there is symmetry about a point, and the origin is placed at this point.

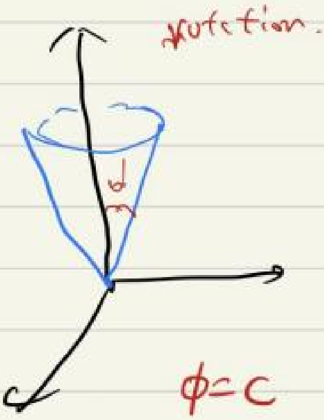
Example.



a sphere



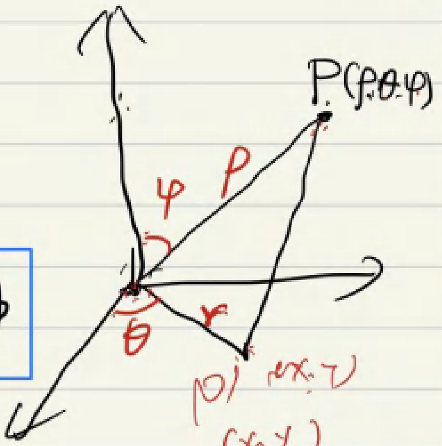
a half-plane



a half-cone

$(\rho, \theta, \phi) \longrightarrow (x, y, z)$

2,



$P'(x, y)$

(x, y)

$= (\rho\sin\phi, \theta)$

$r = \rho\sin\phi \quad z = \rho\cos\phi$

$\Rightarrow$

$$\boxed{x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta \quad z = \rho\cos\phi}$$

Conversely.

$$\boxed{\rho = \sqrt{x^2+y^2+z^2} \qquad \theta = \arctan\frac{y}{x} \qquad \phi = \arccos\frac{z}{\sqrt{x^2+y^2+z^2}}}$$

$\cos\phi = \frac{z}{\rho}$.

Example. spherical coordinates : $(2, \frac{\pi}{4}, \frac{\pi}{3})$

$\quad\rho \quad \theta \quad \phi$

$\Rightarrow$ rectangular coordinates.

$(\sqrt{3}/2, \sqrt{3}/2, 1)$.

$\Delta S \longrightarrow \Delta T$

$T \longrightarrow$

rectangular coordinates $(0, 2\sqrt{3}, -2)$

$r\,dr\,d\theta$

$=$

$\Rightarrow$ spherical coordinates.

$(4, \frac{\pi}{2}, \frac{2\pi}{3})$

Formula for triple integration in spherical coordinates.

☆

$$\iiint_E f(x,y,z)\,dV$$

$$= \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\boxed{\rho^2\sin\phi}\,d\rho\, d\theta\, d\phi$$

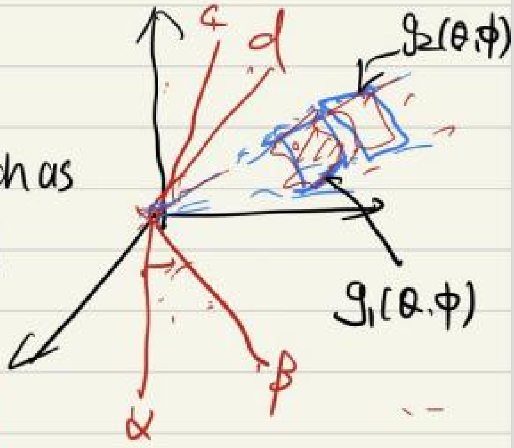
where E is a spherical wedge given by

$$E = \{(\rho, \theta, \phi) \mid \underline{a \le \rho \le b}, \underline{\alpha \le \theta \le \beta}, \underline{c \le \phi \le d}\}$$

"spherical box"

This formula can be extended to include more general spherical regions such as

☆ $E = \{(\rho, \theta, \phi) \mid \alpha \leq \theta \leq \beta,\ c \leq \phi \leq d,\ g_1(\theta, \phi) \leq \rho \leq g_2(\phi, \phi)\}$.



Usually, spherical coordinates are used in triple integrals when surfaces such as cones and spheres form the boundary of the region of integration.

$\theta = c$

$\rho = c$

$(\int_d^c f\ d\rho \ldots)$

Example $\iiint_B e^{(x^2+y^2+z^2)^{3/2}} dV$, B = unit ball. ← bounded by sphere

Solution $B = \{(\rho, \theta, \phi) \mid 0 \leq \rho \leq 1,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \pi\}$.

[illegible] ≤ 1

all directions

$$\iiint_B e^{(x^2+y^2+z^2)^{3/2}} dV$$

$$= \int_0^\pi \int_0^{2\pi} \int_0^1 e^{\rho^3} \boxed{\rho^2 \sin\phi}\, d\rho\, d\theta\, d\phi$$

$$= \underline{\int_0^\pi \sin\phi\, d\phi}, \underline{\int_0^{2\pi} d\theta}\ \underline{\int_0^1 \rho^2 e^{\rho^3}, d\rho}$$

$$= \underline{2} \cdot \underline{2\pi} \cdot \frac{1}{3} e^{\rho^3}\Big|_0^1 = \frac{4}{3}\pi(e-1)$$

□

$\int_0^\pi \sin\phi\, d\phi = -\cos\phi \Big|_0^\pi = 2$

$\rho^2 d\rho = \frac{d\rho^3}{3}$

$$\text{Vol}(E) = \iiint_E dV = \iiint_E 1 \cdot dV \leftarrow \text{Riemam sum} \downarrow \text{limit}$$

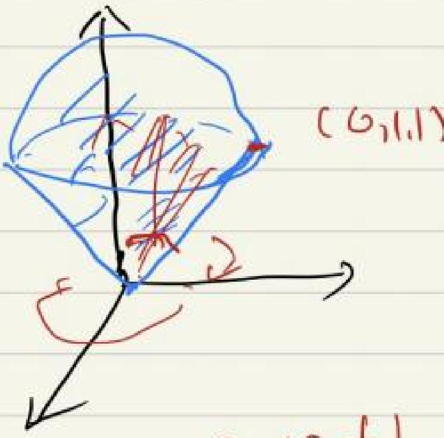
4.

Example Find the volume of the solid that lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = z$.

spherical coordina

Solution $x^2+y^2+z^2 = z \iff x^2+y^2+(z-\frac{1}{2})^2 = \frac{1}{2}$,

center $(0,0,\frac{1}{2})$, radius $\frac{1}{2}$



$x^2+y^2+z^2 = z$

$\iff \rho^2 = \rho\cos\phi$

$\iff \rho = \cos\phi$.

cone : $\phi = \frac{\pi}{4}$. $\Leftarrow$ any half cone.

$\phi = c$.

$\le \rho \le g_2(\theta,\phi)$

$E = \{(\rho,\theta,\phi) \mid 0 \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{4},\ 0 \le \rho \le \cos\phi\}$

$$V(E) = \iiint_E dV = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos\phi} \underline{\rho^2 \sin\phi}\, d\rho\, d\phi\, d\theta.$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} \sin\phi \cdot \frac{\rho^3}{3}\Big|_{\rho=0}^{\rho=\cos\phi} d\phi$$

sinϕ constant w.r.t ρ.

$$= \frac{2\pi}{3} \cdot \int_0^{\frac{\pi}{4}} \sin\phi \cos^3\phi\, d\phi$$

$$= \frac{2\pi}{3}\left[-\cos^4\phi\right]\Big|_0^{\frac{\pi}{4}} = \frac{\pi}{8}.$$

$\sin\phi\, d\phi = -\cos\phi$.

Spherical:

step1: $E = \{(\rho,\theta,\phi) \mid \phi, \theta$. intervals

$g_1(\theta,\phi) \le \rho \le g_2(\theta,\phi)$

step2 $\iiint_E = \int_\rho \int_\theta \int_\phi f \cdot \underline{\rho^2 \sin\phi}\, d\cdots$

2. Change of coordinates in multiple integrals.

why change of coordinates?

difficulitis in computing double integrals:

$$\begin{cases} f(x,y) \text{ or } f(x,y,z) \\ \text{domain } D. \text{ or } E \end{cases}$$

We want to change variables, so that the function or the domain looks simpler.

(polar. coordinates, type 2

$\{(r,\theta) \mid \underline{\alpha\leq\theta\leq\beta,\ h_1(\theta)\leq\rho\leq h_2(\theta)}\}.$

cylindrical coordinates

$\{(r,\theta,z) \mid \alpha\leq\theta\leq\beta,\ h_1(\theta)\leq\rho\leq h_2(\theta),$

$\underline{u_1(r\cos\theta, r\sin\theta)\leq z\leq u_2(r\cos\theta, r\sin\theta)},$

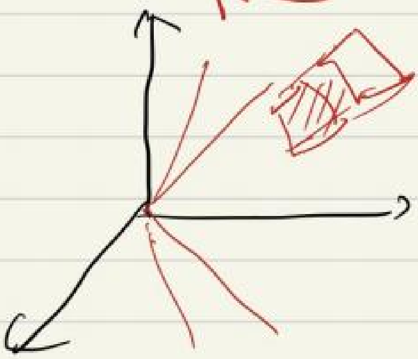
type 2.

spherical coordinates:

$\{(\rho,\theta,\phi) \mid \alpha\leq\theta\leq\beta,\ c\leq\phi\leq d,$

$\underline{g_1(\theta,\phi)\leq\rho\leq g_2(\theta,\phi)}$

type ..



Remark: Single variable, change variable to simplify ~~the~~ functions.

Example $\int \underline{\underline{x}}e^{x^2}dx = \frac{1}{2}e^{x^2}+C$

$x \to x^2$

$\int_a^b \quad \int_c^d$

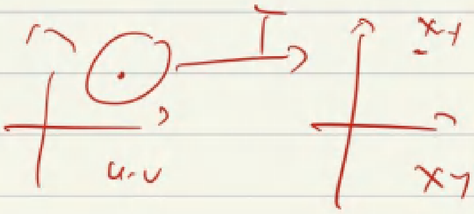
interval

$\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ ordered pair of functions of 2-variables.

Consider a change of variables that is given by a transformation T from the uv-plane to the xy-plane:

$$T(u,v) = (x,y).$$

$x = x(u,v),\ y = (u,v)$



We usually assume that T is C^1.

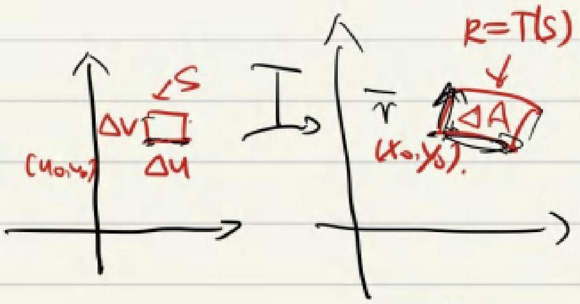
If $T(u_1,v_1) = (x_1,y_1)$, then (x_1,y_1) is called the image of the point (u_1,v_1).

If no two points have the same image, T is called one-to one. (injective).

If T is a one-to one transformation, then it has an inverse transformation T^{-1}. (solve equations).

Now let's see how a change of variables affects a double integral.

Step 1:



Since x, y are C^1, partial derivative

(C')

Scalar equat.

$$x(u_0+\Delta u, v_0+\Delta v) = x(u_0,v_0) + x_u(u_0,v_0)\Delta u + x_v(u_0,v_0)\Delta v + \text{higher order terms},$$

$$y(u_0+\Delta u, v_0+\Delta v) = y(u_0,v_0) + y_u(u_0,v_0)\overset{\Delta u}{} + y_v(u_0,v_0)\Delta v + \text{higher order terms}.$$

Let $\vec{r}(u,v) = (x(u,v), y(u,v))$.

$$\vec{r}_u(u,v) = (x_u(u,v), y_u(u,v))$$

$$\vec{r}_v(u,v) = (x_v(u,v), y_v(u,v)).$$

$$\vec{r}(u_0+\Delta u, v_0+\Delta v) = r(u_0+v_0) + \vec{r}_u \Delta u + \vec{r}_v \Delta v + h.o.t$$

7.

vector equation

$$\Rightarrow \vec{r}(u_0+\Delta u, v_0) - \vec{r}(u_0, v_0) = \vec{r}_u(u_0,v_0)\underline{\Delta u} + \text{higher order terms.}$$

$$\vec{r}(u_0, v_0+\Delta v) - (\vec{r}(u_0,v_0) = \vec{r}_v(u_0,v_0)\Delta v + \text{higher order terms,}$$

when $\Delta u, \Delta v \to 0$.

Area of $R=T(S)$ is the area of the parallelgram spaned by $\underline{\vec{r}_u \cdot \Delta u}$, $\underline{\vec{r}_v \Delta v}$ + higher order terms.

$$\text{Area } R = \Delta u \cdot \Delta v \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_u & y_u & 0 \\ x_v & y_v & 0 \end{vmatrix} = \Delta u \Delta v \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} + \text{higher order terms.}$$

vectors in $\mathbb{R}^3$

($\vec{r}_u, \vec{r}_v$ vectors in $\mathbb{R}^2$, $(x,y) \to (x,y,0)$)

Definition The Jacobian of the transformation T. is,

$$\underline{\frac{\partial(x,y)}{\partial(u,v)}} = \det\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \det \underline{J(T)}. = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$$

a function of 2 variable

$\mathbb{R}^2 \to \mathbb{R}^2$

$$\Rightarrow \Delta A = \underline{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|} \Delta u \Delta v,$$

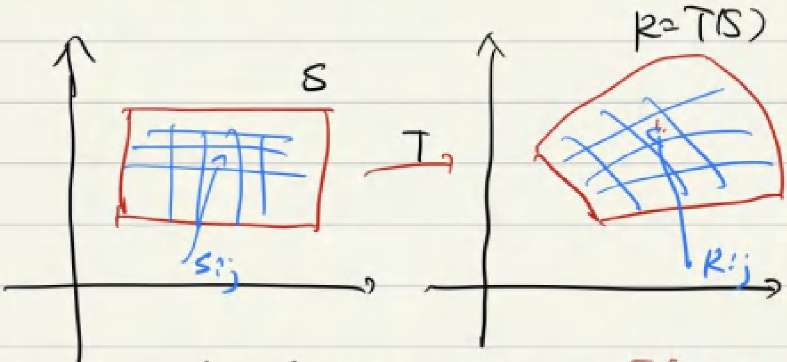
2×2 matrix

(in first order terms).

Step 2

Suppose S is a rectangle,

when we divide S into sufficiently small rectangles s_{ij},



The "diameter" of the division $\{R_{ij}\} \to 0$. $R_{ij} = T(s_{ij})$

$$\text{So } \underbrace{\sum_{i=1}^{m}\sum_{j=1}^{n} f(x,y)\Delta A}_{\iint_R f(x,y)\,dA} = \underbrace{\sum_{i=1}\sum_{j=1}^{n} f(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\Delta u \Delta v}_{f(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv}.$$

$T:$ $S \to R$ $\qquad$ T: 1-1. onto

The limit of the Riemann sum is.

$$\iint_R f(x,y)\,dA = \iint_S f(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$$

We may easily extend to general region S,

Change of variables in a double integral

Suppose that T is a C^1 transformation whose Jacobian is nonzero and that T maps a region S in the uv-plane ✓ onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II regions. Suppose also that T is one-to-one, except perhaps on the boundary of S, then.

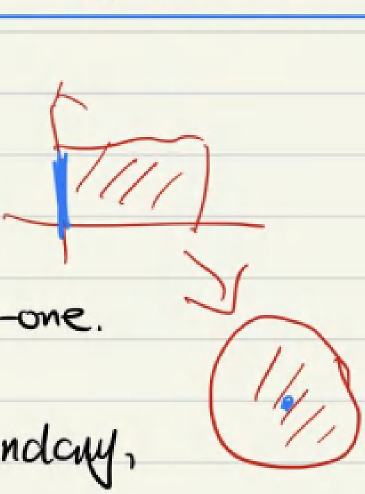
$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$$

✱ absolute value

Remark: $(r,\theta) \xrightarrow{T} (r\cos\theta, \sin\theta)$.

$T(0,\theta) = (0,0)$.

on this boundary, T is NOT one-to-one.



But T is one-to-one outside the boundary,

Remark: $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| > 0$.

but $\int_a^b f(x)\,dx = \int_c^d f(g(u))\,g'(u)\,du$. $\qquad g'(u) < 0$ can happen.

$x = -t$ $\qquad$ (-1)

This is because there is a natural orientation in $\mathbb{R}^1$.

But "orientation" is not used in the definition of multiple integral.

$\int_a^b f(x)dx$ ← sign

$= \lim_{n\to\infty} \sum_i f(x_i^*) \frac{b-a}{n}$

↑

b-a may be <0.

$\iint_D f(x,y)dA$

$= \lim_{m,n\to\infty} \sum_i \sum_j f(x_{ij}^*, y_{ij}^*) \Delta A$

↑

always positive.

(Simplify the domain)

Example $T(u,v) = (u^2-v^2, 2uv)$

$S = [0,1]\times[0,1]$.

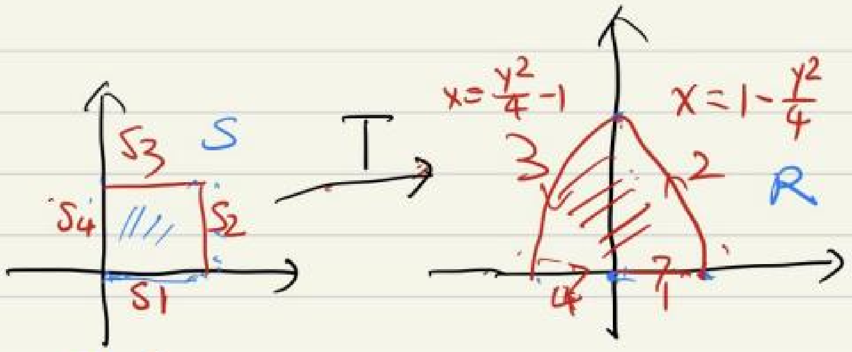


image of boundaries: (☆)

$(0,0) \to (0,0)$. $(1,0) \to (1,0)$. $(1,1) = (0,2)$

$(0,1) = (-1,0)$.

S_1: $(u,0) \to (u^2,0)$. $0\le u\le 1$

$\to$ Image of S_1 = interval $[0,1]$.

parametric curve

S_2: $(1,v) \to \underline{(1-v^2, 2v)\quad 0\le v\le 1}$

$\Rightarrow$ image of S_2: $1-x = \frac{y^2}{4}$. $0\le y\le 2$.

S_3: S_4 ...

↑

boundary ⟷ parametric curves.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = \underline{4(u^2+v^2)}>0.$$

Now we consider

$$\iint_{\underline{\underline{R}}} y\,dA,$$

$$\parallel$$

$x,y = (u^2-v^2, \underline{\underline{2uv}})$

☆

$R \to S.$

$y \to 2uv$

$dA \to \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$

area element

It is

$$\iint_S \underline{2uv} \cdot \underline{4(u^2+v^2)}\,dudv$$

$$= 8 \underline{\int_0^1\int_0^1} u^3v + uv^3\,dudv.$$

(symmetry)

$$= 16 \int_0^1\int_0^1 u^3v\,dudv$$

$$\int_0^1\int_0^1 u^3v\,dudv = \int_0^1\int_0^1 uv^3\,dudv$$

$$= 16 \int_0^1 u^3du \int_0^1 vdv$$

$$= 16 \times \tfrac{1}{4} \times \tfrac{1}{2} = 2.$$ □

(simplify the function and domain)

Example $\iint_R e^{(x+y)/(x-y)}\,dA,$

R: the trapezoidal region with vertices.

(1,0). (2,0) (0,-2), (0,-1)

Solution Make a change of variables:

$u = x+y \quad v = x-y. \longrightarrow$ simplify $e^{\frac{x+y}{x-y}}$

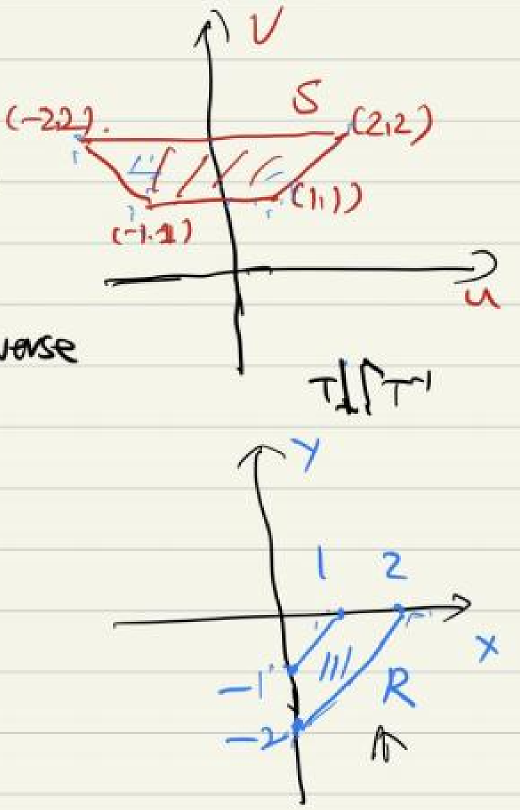
$\leftarrow T^{-1}$

$$\Rightarrow \quad x = \tfrac{1}{2}(u+v), \quad y = \tfrac{1}{2}(u-v) \leftarrow T$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2},$$

$T^{-1}(x,y) = (x+y, x-y)$

To find S in the uv- plane, such that $T(S)=R$, we need to find $T^{-1}(R)$.



But T^{-1} is linear, so we only need to find the inverse image of the four vertices:

$T^{-1}(1,0) = (1,1)$, $T^{-1}(2,0) = (2,2)$

$T^{-1}(0,-2) = (-2,2)$ $T^{-1}(0,-1) = (-1,1)$.

S is a region of type II.

$S = \{(u,v) \mid 1 \le v \le 2,\ -v \le u \le v\}$.

Therefore,

$$\iint_R e^{\frac{x+y}{x-y}}\, dA = \iint_S e^{\frac{u}{v}} \cdot \frac{1}{2}\, du\, dv.$$

$$= \frac{1}{2}\int_1^2 \int_{-v}^{v} e^{\frac{u}{v}}\, du\, dv.$$

R → S,
function
Area element.

$$\int_{-v}^{v} e^{\frac{u}{v}}\, du = v e^{\frac{u}{v}} \Big|_{u=-v}^{u=v} = v(e-e^{-1}).$$

$$\Rightarrow \iint_R e^{\frac{x+y}{x-y}}\, dA = \frac{1}{2}(e-e^{-1}) \int_1^2 v\, dv$$

$$= \frac{3}{4}(e-e^{-1}).$$

$\frac{d}{du} e^{\frac{u}{v}}$
$= \frac{1}{v} e^{\frac{u}{v}}$

What T?

1) simplify domain
function

3, Triple integrals.

Let T be a transformation that maps a region S in uvw-space onto a region R in xyz-space,

The Jacobian of T is just

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det J(T),$$

Then under "good" conditions

$$\iiint_R f(x,y,z)\,dV = \iiint_{\underline{S}} f(x(u,v,w), y(u,v,w), z(u,v,w)) \underline{\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|} du\,dv\,dw.$$

☆

Example Triple integration in spherical coordinates.

Solution $x = \rho\sin\phi\cos\theta,\quad y = \rho\sin\phi\sin\theta,\quad z = \rho\cos\phi$ (spherical coordinates)

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \sin\phi\cos\theta, & -\rho\sin\phi\sin\theta, & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \underline{\cos\phi} & \bigcirc\!\!\!\!\!0 & \underline{-\rho\sin\phi} \end{vmatrix}$$ det 3×3 matrix

expand w.r.t the last row.

$$= \cos\phi.(-\rho^2\sin\phi\cos\phi) - \rho\sin\phi\cdot\rho\sin^2\phi$$

$$= -\rho^2\sin\phi, \qquad (0 \le \phi \le \pi)$$

$$\Rightarrow \left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| = \rho^2\sin\phi. \qquad \square$$

tr

$\iint \quad \iiint$

$\int_C \quad \iint_S$

Lecture 18. vector fields.

- What we know:

$\int \vec{r}(x)\,dx = \langle \int_a^b f(x)\,dx, \int g(x)$

$\int_a^b f(x)\,dx.$

$$\left.\begin{array}{l}\iint_D f(x,y)\,dA \\ \iiint_E f(x,y,z)\,dV\end{array}\right\} \overset{\text{Fubini}}{\Longleftrightarrow} \text{Iterated integrals.}$$

- What we want to learn? ⟵ vector field.

$\int_C f(x,y)\,ds \qquad \int_C \vec{F}\cdot d\vec{r} \to C.$

$C: \mathbb{R} \to \mathbb{R}^3.$ $\longrightarrow$ line integrals $= \int_a^b g(t)\,dt.$
$t \to \vec{r}(t).$

$\iint_S f\,dA \qquad \iint_S \vec{F}\cdot d\vec{S}.$

$S: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ $\longrightarrow$ surface integrals $= \iint_D g(u,v)\,du\,dv.$
$(u,v) \to \vec{r}(u,v).$ ↖parametrization $\qquad D \in \mathbb{R}^2.$

- Question:
Why are there two types of integrals over curves or surfaces?

This can be explained by differential forms.

I will give a short introduction to differential forms on Aug, 19.

ordered pair of scaler fields / ← $(x,y) \in D \to f(x,y))$

2,

1, function of multiple variables $(x,y) \in D \longrightarrow \vec{F}(x,y)$

✓

Definition Let D be a set in $\mathbb{R}^2$. A vector field on $\mathbb{R}^2$ is a function $\vec{F}$ that assigns to each point (x,y) in D a two-dimensional vector $\vec{F}(x,y)$,

since $\vec{F}(x,y)$ is a two-dimensional vector, we can write it in terms of its component functions. P and Q:

$$\vec{F}(x,y) = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$$

$$\vec{F} = P\vec{i} + Q\vec{j}$$

P and Q are scalar functions of two variables and are sometimes called scalar fields. ✓

Definition. Let E be a subset of $\mathbb{R}^3$. A vector field on $\mathbb{R}^3$ is a function $\vec{F}$ that assigns to each point (x,y,z) in E a three-dimensional vector $\vec{F}(x,y,z)$

$$\vec{F}(x,y,z) = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}.$$

It is often useful to picture a vector field by drawing the arrow representing the vector $\vec{F}(x,y)$ starting at the point (x,y).

of course, it is impossible to do this for all points (x,y), but we can gain a reasonable impression of $\vec{F}$ by doing it for a few representative points in D.

$(x,y) \longrightarrow (-y, x).$

Example $\vec{F}(x,y) = -y\vec{i} + x\vec{j}.$

Solution (1).

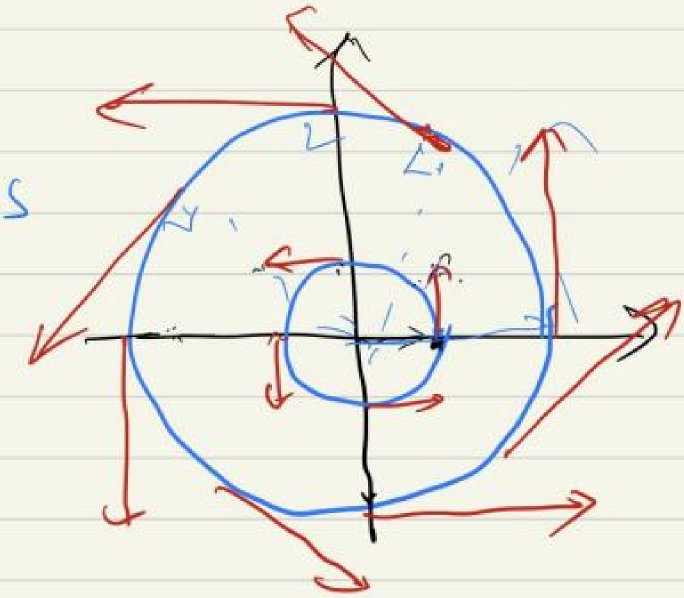
$$\vec{x} \cdot \vec{F}(x) = (x,y) \cdot (-y, x) = 0.$$

$$\|\vec{F}(x,y)\| = \|\vec{x}\| = \sqrt{x^2+y^2}$$

$\vec{F}(x,y)$ is tangent to a circle with center the origin and radius $|\vec{x}|$, The magnitute of the vector is equal to the radius of the circle.

① directions.
magnitudes / lengths

②. a few points.



2, Operations.

We are mainly interested in vector fields on $\mathbb{R}^3$,

① gradient . (scalar fields ⟶ vector fields)

If f is a scalar function of two variables, its gradient ∇f is defined by.

$$\nabla f = f_x(x,y)\vec{i} + f_y(x,y)\vec{j}$$

∇f is a vector field on $\mathbb{R}^2$, and is called the gradient vector field.

Likewise, if f is a scalar function of three variables, its gradient is a vector field on $\mathbb{R}^3$, given by

$$\nabla f(x,y,z) = f_x(x,y,z)\vec{i} + f_y(x,y,z)\vec{j} + f_z(x,y,z)\vec{k}.$$

↑

Example: $f(x,y) = x^2y - y^3.$

$f_x = 2xy, \quad f_y = x^2 - 3y^2.$

$\nabla f(x,y) = 2xy\vec{i} + (x^2-3y^2)\vec{j}$

We introduce the vector differential operator ∇ as.

$$\boxed{\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}.}$$

Not a usual vector.

Just consider it as a vector, whose components are differential operators: $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

consider f as a scalar, then. ∇f is just the "scalar. multipliction": the components of ∇f are "$f \cdot \frac{\partial}{\partial x}$", where the multiplication of a differential operator D is just Df.

$f \mapsto \frac{\partial}{\partial x} f,$

$$f \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(f \cdot \frac{\partial}{\partial x}, f\frac{\partial}{\partial y}, f\frac{\partial}{\partial z}\right)$$

② Curl (vector fields ⟶ vector fields)

If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a vector field,

then the curl of $\vec{F}$ is the vector field on $\mathbb{R}^3$ defined by

$$\boxed{\operatorname{curl}\vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}}$$

Equivalently,

$$\boxed{\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}}$$

∇ : differential - operator-valued vector.

Here, the multiplication of a differential operator D and a function f is just Df.

$D : f \to Df$
function function.

Example $\vec{F}(x,y,z) = xz\,\vec{i} + xyz\,\vec{j} - y^2\vec{k}$,

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(-y^2) - \frac{\partial}{\partial z}(xyz)\right]\vec{i} - \left[\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial z}(xz)\right]\vec{j}$$

$$+ \left[\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(xz)\right]\vec{k}$$

$$= \langle -2y - xy,\ x,\ yz\rangle.$$

③ Divergence (vector fields ⟶ scalar fields)

If $F = P\vec{i} + Q\vec{j} + R\vec{k}$. is a vector field on $\mathbb{R}^3$, then the divergence is the function of three variables defined by

$$\boxed{\operatorname{div}\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}}$$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

Equivalently,

$$\boxed{\operatorname{div}\vec{F} = \nabla\cdot\vec{F}} \swarrow \quad F = (P.\ Q, R)$$

Example $\vec{F}(x,y,z) = \underset{P}{\underline{xz}}\,\vec{i} + \underset{Q}{xyz}\,\vec{j} - \underset{R}{y^2\vec{k}},$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2)$$

$$= z + xz$$

3, Relations between these operations.

$$\begin{array}{ccccccc} \text{scalar field} & \xrightarrow[\nabla]{\text{grad}} & \text{vector field} & \xrightarrow[\nabla\times]{\text{curl}} & \text{vector field} & \xrightarrow[\nabla\cdot]{\text{div}} & \text{scalar field} \\ f & & \vec{F} & & \vec{F} & & f \end{array}$$

①

Theorem:

The compositions of two adjacent operations are zero.

(1)

$$\text{curl}(\nabla f) = 0.$$

Proof:

$$\text{curl}(\nabla f) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

mixed differential theorem.

$$= \left(\underline{\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}}\right)\vec{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\vec{j}$$

$0 =$

$$+ \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\vec{k} \quad = \vec{0}.$$

(2) $\text{div curl} \vec{F} = 0.$ $\quad \vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

$$\text{div curl} \vec{F}$$

$$= \frac{\partial}{\partial x}\left(\underline{\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}}\right) + \frac{\partial}{\partial y}\left(\underline{\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}}\right) + \frac{\partial}{\partial z}\left(\underline{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}\right).$$

$$= \underline{\frac{\partial^2 R}{\partial x \partial y}} - \underline{\frac{\partial^2 Q}{\partial x \partial z}} + \underline{\frac{\partial^2 P}{\partial y \partial z}} - \underline{\frac{\partial^2 R}{\partial y \partial x}} + \underline{\frac{\partial^2 Q}{\partial z \partial x}} - \underline{\frac{\partial^2 P}{\partial z \partial y}} = 0.$$

Example $\vec{F}(x,y,z) = xz\,\vec{i} + xyz\,\vec{j} - y^2\vec{k}.$

$$\text{curl} \vec{F} = -y(2+x)\vec{i} + x\vec{j} + yz\vec{k} \neq \vec{0}.$$

So it is NOT ∇f. for any f.

$$\text{div} \vec{F} = z + xz \neq 0.$$

So $\vec{F}$ is NOT curl $\vec{G}$. for any $\vec{G}$,

$$\begin{tikzcd} \nabla\cdot, \arrow[r, bend right] \arrow[drr, bend right] & \nabla\times F^{0} \arrow[r, bend right] & \nabla\cdot \\ & & \nabla^2=\Delta \end{tikzcd}$$

② Laplace operator. (scalar fields to scalar fields).

However.

$$\boxed{\operatorname{div}(\nabla f) = \nabla\cdot(\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}}$$

We often write it as $\nabla^2 f$ or Δf. ← PDE

∇ is $\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$.

the dot product of ∇ is.

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

and $\nabla^2 f$ is exactly $\operatorname{div}(\nabla f)$.

$\nabla^2\vec{F}$ is defined to be $\nabla^2\vec{F} = (\nabla^2 P, \nabla^2 Q, \nabla^2 R)$.

Remark A function f satifies the differential equation.

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

is called a harmonic function.

Harmonic Functions play an essential role in many branches of mathmatics and physics (complex analysis, partial differential equation, etc).

Question 1:

why these operators? why $\text{curl} \circ \nabla = \text{div} \cdot \text{curl} = 0$?

Answer:

differential forms!

operations on vector fields
$\Updownarrow$
exterior differentials of differential forms

Question 2:

Geometric meaning of these operators?

Gradient: direction that changes fast:

$D_{\vec{u}} f$ attains its maximal in the direction of ∇f.

curl, div: we will discuss later.

$\square$

4. Conservative vector fields

Definition A vector field $\vec{F}$ is called a conservative vector field if theres exists a function f, such that

$\vec{F} = \nabla f$, In this situation, f is called a potential function for $\vec{F}$.

Remark Potential functions are not unique: If $\vec{F} = \nabla f$,

$\vec{F} = \nabla (f + c)$ c a constant. $\nabla c = 0$.

However, If $\vec{F} = \nabla f = \nabla g$, $\Rightarrow \nabla (f-g) = 0$. $(f-g)_{x,y,z} = 0$

$f-g$ should be (locally constant).

"unique" in some sense : up to a constant curl $\cdot \nabla f = 0$

- A necessary condition:

  If $\vec{F}$ is a conservative vector field,

  then curl $\vec{F} = \vec{0}$.

$\frac{\partial}{\partial x} f = P$

$\frac{\partial}{\partial y} f = Q$

$\frac{\partial}{\partial z} f = R$

Question. Is this also a sufficient condition?

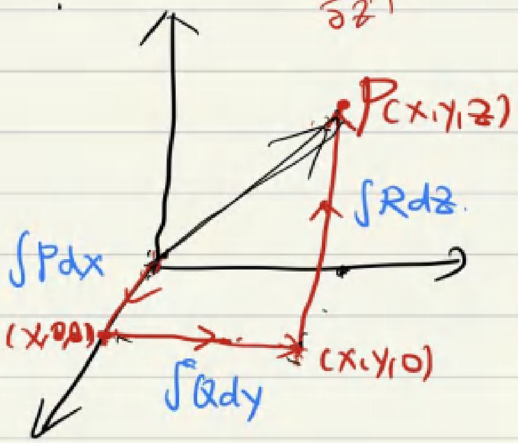
Answer. Yes, locally, No, globally, ✻

✻ If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k} = \nabla f$.

Let $f(0)=0$. , P be an arbitrary point

Integrate with respect to x.

we get $f(x,0,0)$. $P = \frac{\partial f}{\partial x}$;



Integrate with respect to y,

we get $f(x,y,0)$,

Integrate with respect to z, we get $f(x,y,z)$.

(If f exists)

So f is uniquely determined if the value of f at a point (say, the origin) is specified.

However, we may get $f(x,y,z)$ by integrating along any curves starting from 0, ending at P. Do they give the same result?

Green's

curl $\vec{F}=0 \Rightarrow$ yes, locally,

But to find a global f, we must consider topology.

simply connected.

A special case:

If $\vec{F}$ is defined on all of $\mathbb{R}^3$, then

$$\boxed{\text{curl } \vec{F} = 0 \iff \text{conservative}} \iff \vec{F} = \nabla f$$

Example $\vec{F}(x,y,z) = y^2z^3\,\vec{i} + 2xyz^3\,\vec{j} + 3xy^2z^2\,\vec{k}$.

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$$

$\vec{F}$ defined on all the $\mathbb{R}^3$

$$= (6xyz^2 - 6xyz^2)\vec{i} - (3y^2z^2 - 3y^2z^2)\,\vec{j} + (2yz^3 - 2yz^3)\,\vec{k} = \vec{0}.$$

$\vec{F} = \nabla f.$

To find a funcionf, we assume $f(0) = 0$.

Let $P = (x_0, y_0, z_0)$.

Then $f(x_0,0,0)$

$$= \int_0^{x_0} f_x(x,y,z)\,dx = \int_0^{x_0} 0\,dx = 0.$$

$= y^2z^3$ $(z=0.\ y=0.)$ $0 \le x \le x_0$

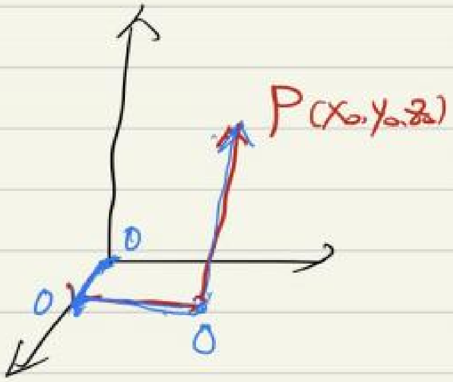
$$f(x_0,y_0,0) = 0 + \int_0^{y_0} f_y(x,y,z)\,dy \to 2xyz^3 = 0$$

$$= \int_0^{y_0} 0\,dy = 0.$$

$$f(x_0,y_0,z_0) = 0 + \int_0^{z_0} f_z(x,y,z)\,dz$$

$$= \int_0^{z_0} 3\underline{x_0y_0^2}\,z^2\,dz$$

$$= x_0y_0^2\,z^3\Big|_{z=0}^{z_0} = x_0y_0^2z_0^3$$



P (x_0,y_0,z_0) is arbitrary, so $f = \underbrace{xy^2z^3}_{} + K$ (constant)

Easy to check that $\vec{F} = \nabla f$. □

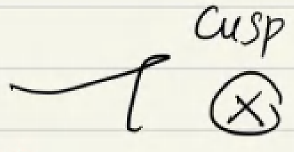
Lecture19. Line integrals

$\int_a^b f(x)dx$ $\Downarrow$ $\Uparrow$

$\begin{cases} f(x_i^*) \longrightarrow f(x_i^*, y_i^*) \\ \Delta x_i \longrightarrow \underline{\underline{\Delta s_i}} \end{cases}$

$\int_C f(x,y)ds$ $\qquad$ $\int_C f(x,y,z)ds.$

$\int_C \vec{F}(x,y)d\vec{r}$ $\qquad$ $\int_C \vec{F}(x,y,z)d\vec{r}.$



arc length

plane curve (ds)

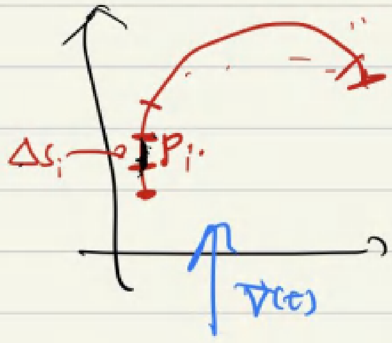
1. We start with a plane curve C given by the parametric equations

$$x=x(t) \quad y=y(t), \qquad a\le t\le b. \ , \quad \text{or}\ \vec{r}(t)=x(t)\vec{i}+y(t)\vec{j}$$

and we assume that C is a smooth curve. ($\vec{r}'(t)\neq\vec{0}$)

Divide the parameter interval [a,b] into n subintervals $[t_{i-1}, t_i]$ of equal width. Let $x_i=x(t_i)$, $y_i=y(t_i)$. Then $P_i = (x_i, y_i)$ divide C into n subarcs with lengths Δs_i.

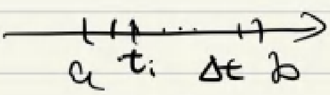
Choose any point $P_i(x_i^*, y_i^*)$ in the i-th subarc. (sample points).



⇒ Riemann sum

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\underline{\Delta s_i}$$

$\Big\{$ limit $\Sigma \to \int$.



Definition. If f is defined on a smooth curve C given by parametric equations: $x=x(t)$ $y=y(t)$ $a\le t\le b$, then the line integral of f along C is

$$\int_C f(x,y)ds = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

if this limit exists.

when $n\to\infty$.

$ds = |\vec{r}'(t)|\,dt$ + higher order terms.

So we get $\qquad f(x,y) = f(x(t)\ y(t))$.

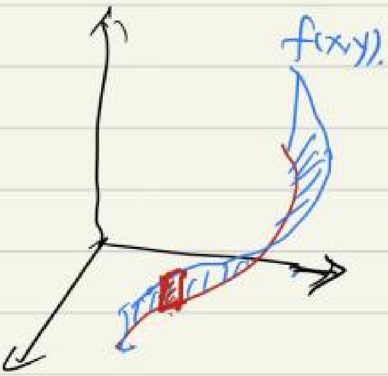
$$\boxed{\begin{aligned}\int_C f(x,y)\,ds &= \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt\\ &= \int_a^b f(x(t),y(t))\,|\vec{r}'(t)|\,dt\end{aligned}}$$ ⇐☆

☆

Remark: The value of the line integral does not depend on the parametrization of the curve, provided that the curve is traversed $\boxed{\text{exactly once}}$ as t increases from a to b.

☆

Proof: change of variable.



Remark: if $f(x,y)\geq 0$.

$\int_C f(x,y)\,ds$ represents the area of one side of the "fence", whose base is C. and whose height above the point (x,y) is f(x,y).

≈ "Rieman sum" → limit.

$\theta! \longrightarrow \cos\theta, \sin\theta$

$\int_a^{2\pi} \qquad \theta\in[0,2\pi) \xrightarrow{f} S^1.$

$[0, 4\pi] \longrightarrow S^1$ twice

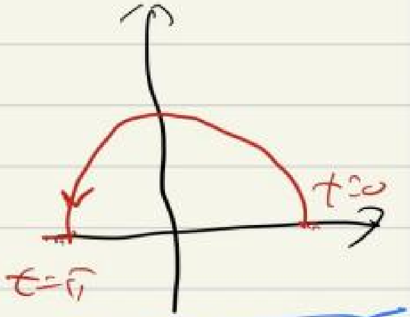
$\int_0^{4\pi} = 2\int_0^{2\pi}$

Example: $\int_C (2+x^2y)\,ds.$

$C :=$ the upper half of the unit circle $x^2+y^2=1$.

Solution: Step 1: parametric equation of C.

C: $x=\cos t$, $y=\sin t$, $0\le t\le\pi$.



$$ds/dt = \sqrt{x'(t)^2+y'(t)^2}$$
$$= \sqrt{\sin^2 t+\cos^2 t} = 1.$$

$$\int_C (2+x^2y)\,ds = \int_0^{\pi} (2+\cos^2 t\sin t)\,dt.$$

$f(x,y) \longrightarrow f(t)$.
$ds \longrightarrow |r'(t)|\,dt$
$\int_C \longrightarrow \int_a^b$

$$\int_0^{\pi} 2\,dt = 2\pi.$$

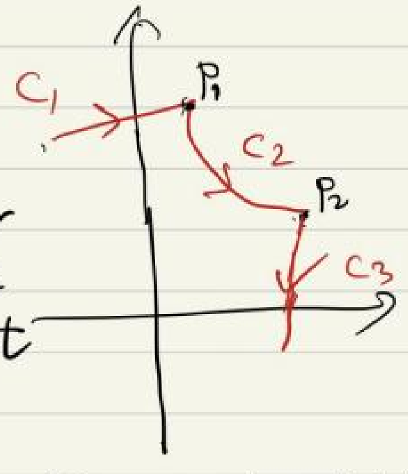
$= -d(\cos)$

$$\int_0^{\pi} \cos^2 t\,\underbrace{\sin t\,dt} = -\frac{\cos^3 t}{3}\Big|_0^{\pi} = \frac{2}{3}.$$

$$\underline{\int_C (2+x^2y)\,ds = 2\pi+\frac{2}{3}.}$$

□

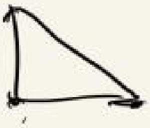
Suppose now that C is a piecewise-smooth curve; that is, C is a union of [illegible] of finite number of smooth curves $C_1 \cdots C_n$, where the initial point of C_{i+1} is the terminal point of C_i.



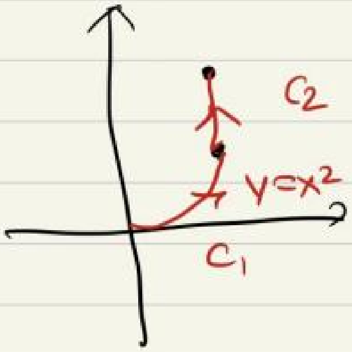
Definition $$\int_C f(x,y)\,ds = \sum_{i=1}^{n} \int_{C_i} f(x,y)\,ds.$$

singular points are ignored

(discrete set)



Example $\int_C 2x\,ds$.



$C = C_1 \cup C_2$:

C_1: $y = x^2$, $0 \leq x \leq 1$.

C_2: the vertical line segment from $(1,1)$ to $(2,1)$

Solution

$$\int_C 2x\,ds = \int_{C_1} 2x\,ds + \int_{C_2} 2x\,ds.$$

C_1 is the graph of the function $y = x^2$, so we have a natural parametrization.

$$\underline{x=x} \quad y = x^2 \quad 0 \leq x \leq 1.$$

$\int_C \to \int_a^b$

$f(x,y) \to f(x)$

$ds \to |\vec{r}'(x)|dx$

$$ds/dx = \sqrt{1+(2x)^2} = \sqrt{1+4x^2}$$

$2x\,dx = d(x^2)$

$$\int_{C_1} 2x\,ds = \int_0^1 2x\sqrt{1+4\underline{x^2}}\,dx = \frac{1}{4}\cdot\frac{2}{3}(1+4x^2)^{3/2}\Big|_0^1 = \frac{5\sqrt{5}-1}{6}$$

$\int x^\alpha = \frac{x^{\alpha+1}}{\alpha+1}$

On C_2, we chose y as the parameter.

$$\underline{x=1} \quad y=y \quad 1 \leq y \leq 2. \qquad ds/dy = 1.$$

$= \sqrt{0^2+1^2} = 1.$

$$\int_{C_2} 2x\,ds = \int_1^2 2\,dy = 2.$$

$$\Rightarrow \int_C 2x\,ds = \frac{5\sqrt{5}-1}{6} + 2.$$

□

2, Line integrals of f along C with respect to $\underline{x}$ and $\underline{y}$

replace $\underline{\Delta s_i}$ by $\Delta x_i = x_i - x_{i-1}$.

$$\int_C f(x,y)\underline{dx} = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta x_i$$

$\frac{dx}{dt} = x'(t)$

$\Rightarrow d\cancel{x} = x'(t)\,dt$

Since $\Delta x_i = x'(t_i)\Delta t$ + higher order terms.

$$\boxed{\int_C f(x,y)\,dx = \int_a^b f(x(t), y(t))\underline{x'(t)}\,dt}$$

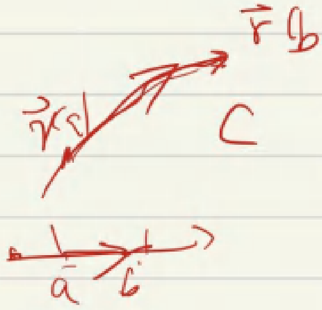
$|r'(t)|$

$\downarrow$

$x'(t)$,

Similarly, we define $\int_C f(x,y)dy$, and

$$\boxed{\int_C f(x,y)\,dy = \int_a^b f(x(t), y(t))\, y'(t)\,dt}$$



1.

Remark ✱

A given parametrization $x = x(t)$, $y = y(t)$, $a \le t \le b$, determines an $\underline{\text{orientation}}$ of a curve C, with the positive direction corresponding to increasing values of the parameter t.

If $-C$ denotes the curve consisting of the same point as C. but with the opposite orientation, then we have

$$\int_{-C} f(x,y)dx = -\int_C f(x,y)dx \qquad \int_{-C} f(x,y)dy = -\int_C f(x,y)dy$$

change of variable

But $\int_{-C} f(x,y)ds = \bigcirc\int_C f(x,y)ds$.

$\int_{-C} = \int_b^a$

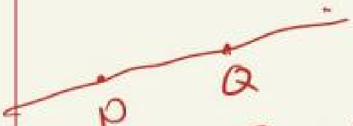
$\int_C = \int_a^b$

$dx = x'(t)dt$, $dy = y'(t)dt$. but $ds = \underline{|\vec r'(t)|}\,dt$ □

$\wedge$

$x \to a+b-t$

>0

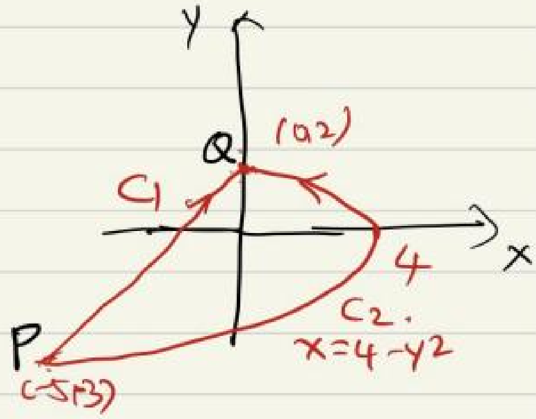


$(\vec{p}+t(\vec{Q}-\vec{p}))$ $\quad \vec{P}+t(\vec{Q}-\vec{P})$ $\quad \vec{P}+t(\vec{Q}-\vec{P})$

$-\infty \le t \le \infty$; $0 \le t \le 1$

$t=0$ P
$t=1$ = Q

Example $\int_C y^2dx + xdy$. $\begin{cases} t=0 \to P. \\ t=1 \to Q \end{cases}$



(a) $C = C_1$

the line segment from (-5,-3) to (0,2).

parametric representation.

$x=5t-5 \quad y=5t-3 \qquad 0\le t\le 1.$

$\vec{Q}-\vec{P} = (5,5)$. $P=(-5,-3)$.

$(5,-3)+t(5,5)$

$dx=5dt \qquad dy=5dt$

$$\int_{C_1} y^2dx+xdy = \int_0^1 (5t-3)^2\cdot 5dt + (5t-5)\cdot 5dt$$

$\int_C \to \int_a^b$

$$= 5\int_0^1 (25t^2-25t+4)dt$$

$\cancel{f(x,y)} \to f(t)$

$\begin{cases} dx \to x'(t)dt \\ dy \to y'(t)dt \end{cases}$

$$= 5\left[\frac{25t^3}{3} - \frac{25t^2}{2} + 4t\right]_0^1 = -\frac{5}{6}$$

(b) $C=C_2$: the arc of the parabola $x = 4-y^2$ from (-5,-3) to (0,2).

parametric equations: graph: $x=4-\underset{\uparrow}{y}^2$

free variable

$x=4-y^2, \quad y=y \qquad -3\le y\le 2$

$dx=-2ydy$

$$\int_{C_2} y^2dx+xdy = \int_{-3}^{2} y^2(-2y)\cdot dy + (4-y^2)dy$$

$$= \int_{-3}^{2} (-2y^3-y^2+4)dy$$

$$= \left[-\frac{y^4}{2} - \frac{y^3}{3} + 4y\right]_{-3}^{2} = 40\frac{5}{6}.$$ □

C: x(t) y(t)

3, Line integrals in Space:

C: $x = x(t)$, $y = y(t)$, $z = z(t)$.

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}.$$

$$f(x(t), y(t)) \longrightarrow f(x(t), y(t), z(t)).$$

$$ds = |\vec{r}'(t)|\,dt \longrightarrow ds = |\vec{r}'(t)|\,dt.$$

$\sqrt{x'(t)^2 + y'(t)^2}$ $\quad$ $\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$

$$dx = x'(t)\,dt \qquad dy = y'(t)\,dt \longrightarrow dx = x'(t)\,dt \quad dy = y'(t)\,dt \quad dz = z'(t)\,dt$$

$$\boxed{\int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t))\,|\vec{r}'(t)|\,dt}$$

$$\boxed{\int_C f(x,y,z)\,dx = \int_a^b f(x(t), y(t), z(t))\,x'(t)\,dt}$$

Integrals along piecewise smooth curves.

orientation and line integrals.

$\uparrow$

ds.
does not chg

dx dy dz

depends on the orientation.

$\int P ds \vec{i} + \int Q ds \vec{j} + \int R ds \vec{k}$ $\vec{F} \longrightarrow \vec{F}|_C = P\vec{i} + Q\vec{j} + R\vec{i}$

$(x(t), y(t), z(t)) \in C$

$\Rightarrow \vec{F}(x(t), y(t), z(t))$

4 Line integrals of vector fields.

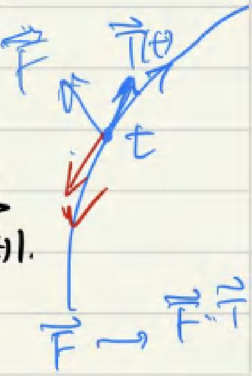
a constant ← NOT vector

Let $\vec{F} = \vec{F}(x, y, z)$ be a vector field on $\mathbb{R}^3$.

If a curve C is given by the vector equation.

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}.$$

We may define unit tangent vectors $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$.



$\vec{F}(x(t), y(t), z(t)) \cdot \vec{T}(t)$. is a function on C,

Then the line integral of $\vec{F}$ along C is defined to be the integration of $\vec{F}\cdot\vec{T}$ along C, (w.r.t ds) $\int \vec{F}\cdot d\vec{r}$. notation.

$$\int_C \vec{F}(\vec{r}(t)), \vec{T}(t)\, ds = \int_C \vec{F}(\vec{r}(t)) \cdot \frac{r'(t)}{|r'(t)|} \cdot |r'(t)|\, dt$$

physics

$$= \int_C \vec{F}(\vec{r}(t)) \cdot r'(t)\, dt.$$

vector-valued func'n

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt}$$

function

Remark. If the orientation of a curve is reversed, $\vec{T}(t) \longrightarrow -\vec{T}(t)$,

$\vec{F}\cdot\vec{T} \longrightarrow -\vec{F}\cdot\vec{T}$, $\Rightarrow$ line integral changes the sign ◻

Remark: If $\vec{F}$ is a vector field on $\mathbb{R}^2$, C is a plane curve. the definition and computation of line integral do not change.

$\vec{r} = (x, y, z)$

$d\vec{r} = (dx, dy, dz)$

Remark. If we write $d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$

$$\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}.$$

Then

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \underline{P\,dx} + Q\,dy + R\,dz$$

$$= \int_a^b \left(\underline{P\,x'(t) + Q\,y'(t) + R\,z'(t)}\right) dt$$

$$= \int_C (\vec{F} \cdot \vec{T})\, ds.$$

This explains the notation $\int_C \vec{F} \cdot d\vec{r}$.

This also explains why integral of vector fields depends on the orientation of curves.

Example $\int_C \vec{F} \cdot d\vec{r}$. $\quad \vec{F}(x,y,z) = \underset{P}{xy}\,\vec{i} + \underset{Q}{yz}\,\vec{j} + \underset{R}{zx}\,\vec{k}$.

C: the twisted cubic.

$x = t, \quad y = t^2, \quad z = t^3 \qquad 0 \le t \le 1.$

Solution $\underline{\vec{r}'(t) = (1, 2t, 3t^2)}$. $\quad \vec{F}(\vec{r}(t)) = \underline{(t^3, t^5, t^4)}$.

$xy = t \cdot t^2$

$yz = t^2 \cdot t^3 = t^5$

$zx = t^4$.

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \underline{\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)}\, dt$$

$$= \int_0^1 (t^3 + 5t^6)\, dt = \frac{t^4}{4} + \frac{5t^7}{7} \Big|_0^1 = \frac{27}{28}. \qquad \square$$

5. The Fundamental Theorem for Line Integrals.

Fundamental theorem of Calculus:

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

$$\begin{cases} \text{interval } [a,b] \longrightarrow \vec{r}(t). \quad a \le t \le b \quad C. \\ F'(x) \longrightarrow \nabla f \quad \nabla f = (f_x, f_y, f_z) \\ dx \longrightarrow d\vec{r} \quad (dx, dy, dz). \end{cases}$$

Theorem Let C be a smooth curve given by the vector function $\vec{r}(t)$, $a \le t \le b$, let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$

Proof:

$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt \qquad t \to (x, y, z) \to f.$$

$$= \int_a^b \left(\frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt} \right) dt$$

$$= \int_a^b \underbrace{f(\vec{r}(t))}_{\mathbb{R} \to \mathbb{R}}\,dt \overset{FTC}{=} f(\vec{r}(b)) - f(\vec{r}(a)). \qquad \square$$

Recall that a vector field $\vec{F}$ is called conservative if $\vec{F} = \nabla f$ for some f.

The line integral of a conservative vector field depends only on the initial point and terminal point of a curve, conservative $\Longleftrightarrow$ independent of path.

If $\vec{F}$ is a continuous vector field with domain D, we say that the line integral $\int_C \vec{F}\cdot d\vec{r}$ is independent of path. if $\int_{C_1} \vec{F} d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$ for any two paths C_1 and C_2 in D that have the same initial points and terminal points.

Example conservative fields. $\Rightarrow$ ind. of path (independent of the "how a particle goes from one place to another)

A curve is called **closed** if its terminal point coincides with its initial point. (initial and final position)

Theorem $\int_C \vec{F}\cdot d\vec{r}$ is independent of path in D if and only if $\int_C \vec{F}\cdot d\vec{r} = 0$ for every closed path C in D.

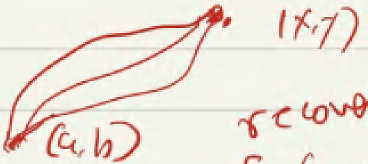
Theorem Suppose $\vec{F}$ is a vector field that is continuous on an open connected region. If $\int_C \vec{F}\cdot d\vec{r}$ is independent of path in D, then $\vec{F}$ is a conservative vector field on D,

indep -- path $\Rightarrow$ conservative

Sketch of proof:

Let A(a,b) be a fixed point.

Define $f(x,y) = \int_C \vec{F}\cdot d\vec{r}$.



recover f from $\vec{F}$

where C is a curve going from (a,b) to (x,y).

f(x,y) is well-defind. Then check that $\nabla f = \vec{F}$ □.

Question How to show $\int_C \vec{F}\cdot d\vec{r} = 0$ for closed curves C?

Green's theorem $\nwarrow$ Tomorrow! ✓

$\int_a^b$ = ← Fundamental thm of Calculus

← $\iint_D$

Lecture 20. Green's theorem.

1. Some topological preparations

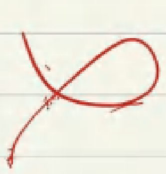
$C: \vec{r}(t) \quad a \le t \le b$

A curve is called closed if its terminal point coincides with its initial point, that is, $\vec{r}(b) = \vec{r}(a)$.

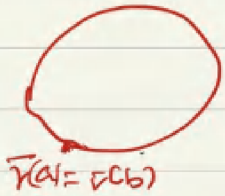
A simple curve C is a curve that doesn't intersect itself anywhere between its endpoints.

→ if: $a < t_1 < t_2 < b$ then $\vec{r}(t_1) \ne \vec{r}(t_2)$

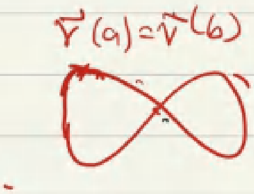
Simple
NOT closed.



not simple
not closed



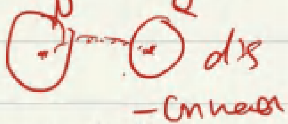
simple
closed.



not simple
closed

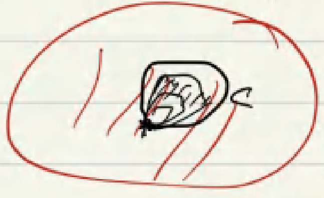
A region D is open if, for every point P in D there is a disk with center P that lies entirely in D. open disks

D is called connected if any two points in D can be joined by a path that lies in D.

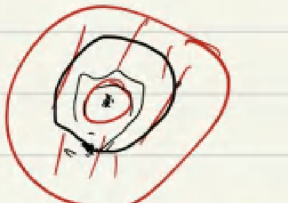


Remark This is actually the definition of path-connected domains, but for open subset in $\mathbb{R}^2$, connected ⇔ path-connected.

A simply connected region in the plane is a connected region D such that every simple closed curves in D encloses only points that are in D.



Simply connected



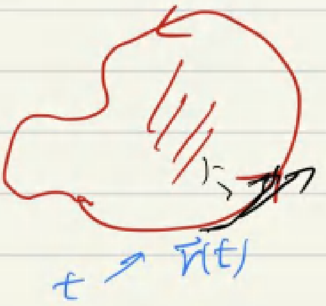
"hole"

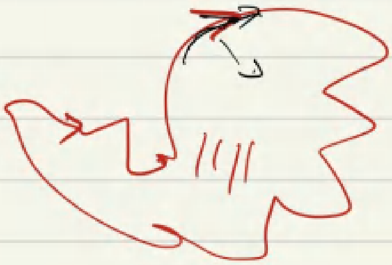
NOT simply connected

2. Green's theorem

line integral around a simple closed curve C ⟺ double integral over the plane region D bounded by C

The positive orientation of a simple closed curve C refers to a single counterclockwise traversal of C. If C is given by the vector function $\vec{r}(t)$, $a \le t \le b$, then the region D is always on the left as the point $\vec{r}(t)$ traverses C.





positive orientation

negative orientation

circles triangles

Green's theorem.
Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Notation $\oint P\,dx + Q\,dy$ is sometimes used to indicate that the line integral is calculated using the positive orientation of the closed curve C.

①

Proof:

For simplicity, we assume that D is both type I and type II. We call such regions simple regions.

We only need to prove that

$$\int_C P\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA.$$

and $$\int_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA.$$

to kill "∂y"

$\int \frac{\partial P}{\partial x} dy = \int$

Write D as

$$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}.$$

FTC.

Then

$$\iint_D \frac{\partial P}{\partial y}\,dA = \int_a^b \underbrace{\int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}\,dy}\,dx = \int_a^b P(x, g_2(x)) - P(x, g_1(x))\,dx$$

$x = x.\ y = g_2(x)$

$x = x.\ y = g_1(x).$

$y = g_1(x)$

$$\int_{C_1} P(x,y)\,dx = \int_a^b P(x, g_1(x))\,dx.$$

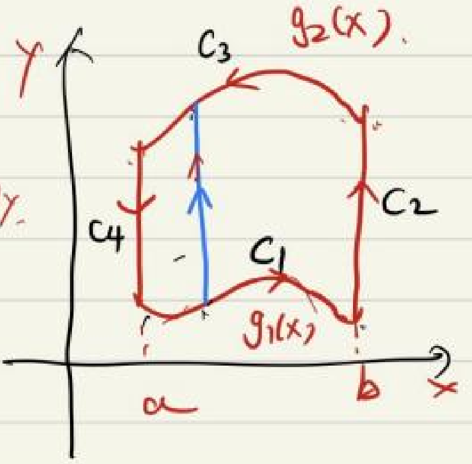
$y = g_2(x)$

$$\int_{C_3} P(x,y)\,dx = -\int_a^b P(x, g_2(x))\,dx$$

$dx = 0.\ dy.$

$$\int_{C_2} P(x,y)\,dx = \int_{C_4} P(x,y) = 0.$$ $a \le y \le b$

$x -$



$$\Rightarrow \int_C P(x,y)\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA.$$

Similarly, $$\int_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA$$

$$\int_{C_3} = \int_{-C_3} = -\int$$

3. Applications

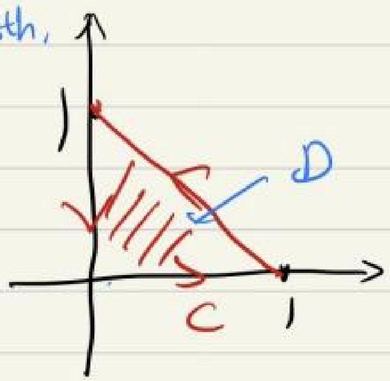
(1) $\int_C P\,dx + Q\,dy$, if ① $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$ are simple, $\int_{C_1} + \int_{C_2} + \int_{C_3}$

we may compute $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$. $\int_0^b \int_0^{2x} \leftarrow$

Or if the domain ② is simple.
(to avoid computing line integrals along piecewise smooth curves)

boundary is smooth, piecewise

Example $\int_C x^4\,dx + xy\,dy$



Solution. $P = x^4$. $Q = xy$

$\frac{\partial P}{\partial y} = 0$ $\frac{\partial Q}{\partial x} = y$.

$$\int_C x^4\,dx + xy\,dy = \iint_D y\,dA$$
$$= \int_0^1 \int_0^{1-x} y\,dy\,dx = \frac{1}{2}\int_0^1 (1-x)^2\,dx$$
$$= -\frac{1}{6}(1-x)^3\Big|_0^1 = \frac{1}{6}$$

Example ✓ $\int_C (3y - \underbrace{e^{\sin x}}_{P})\,dx + (7x + \underbrace{\sqrt{y^4+1}}_{Q})\,dy$

$C = x^2+y^2 = 9$. $D = \{(x,y) \mid x^2+y^2 \le 9\}$.

Solution $P = 3y - e^{\sin x}$, $Q = 7x + \sqrt{y^4+1}$. $\partial P/\partial y = 3$. $\frac{\partial Q}{\partial x} = 7$.

The line integral $= \iint_D 4\,dA = 4\,\text{Area}(D) = 36\pi$

$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$ $4 \cdot \pi r^3$

r^2

$\int_C \longrightarrow \iint$

5,

P, Q.

$1 = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

$\begin{cases} P=0 & Q=x \\ P=-y & Q=0 \end{cases}$

(2)

Area of D is $\iint_D 1\,dA$.

By Green's theorem,

$$\boxed{A = \int_C x\,dy = -\int_C y\,dx = \frac{1}{2}\int_C x\,dy - y\,dx}$$

So we write the area as a line integral,

Remark Generally, $\iint f\,dA$, it is not so easy to find simple P, Q such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = f$.

HW. 4. P. 1.

$\begin{cases} x = a r\cos\theta \\ y = b r\sin\theta \end{cases}$

$\begin{cases} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{cases}$

Example $D = \{(x,y) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1\}$ Find Area(D).

Solution Let $C = \partial D = \{(x,y) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$.

The ellipse has parametric equations.

$x = a\cos\theta \quad y = b\sin\theta, \quad 0 \le \theta \le 2\pi.$

$dy = b\cos\theta\,d\theta \quad dx = -a\sin\theta\,d\theta$

$$A = \frac{1}{2}\int_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} a\cos\theta\, b\cos\theta\,d\theta - b\sin\theta(-a\sin\theta)\,d\theta$$

$$= \frac{ab}{2}\int_0^{2\pi} d\theta = \pi ab, \qquad \cos^2\theta + \sin^2\theta = 1$$

Or. $A = \int_C x\,dy = \int_0^{2\pi} ab\cos^2\theta\,d\theta = \pi ab.$

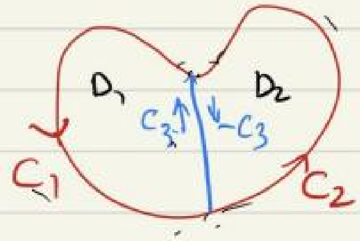
$$\left(\int_0^{2\pi} \cos^2\theta\,d\theta = \int_0^{2\pi} \frac{1+\cos 2\theta}{2}\,d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{2}\Big|_0^{2\pi} = \pi \right)$$

(geometry, topology).

4. Extended versions of Green's theorem

①

D is simple → D is a finite union of simple regions
↓
both type I and II



If $D = D_1 \cup D_2$, D_i simple
$\partial D_1 = C_1 \cup C_3$ $\quad \partial D_2 = C_2 \cup (-C_3)$.

$$\int_{C_1 \cup C_3} Pdx + Qdy = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$ D_1 simple.

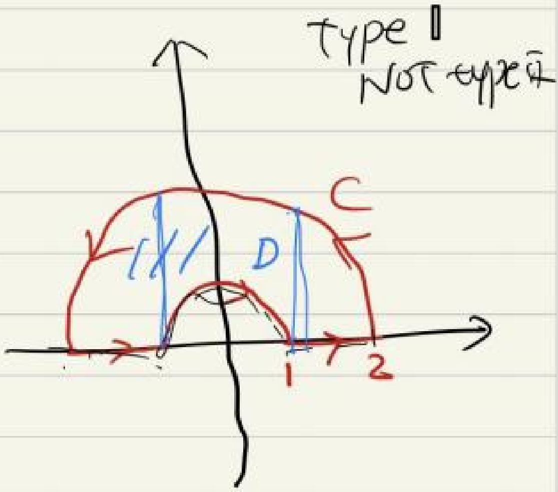
$$\int_{C_2 \cup (-C_3)} Pdx + Qdy = \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$ D_2 simple

Taking sums of both sides.

$$\int_{\underset{= C_1 \cup C_2}{\partial D}} Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Example $\int_C y^2 dx + 3xy\,dy$

Solution $D = \{(r,\theta) \mid 1 \le r \le 2,\ 0 \le \theta \le \pi\}$.



$P = y^2 \quad Q = 3xy \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y$

$$\int_C y^2 dx + 3xy\,dy = \iint_D y\,dA$$

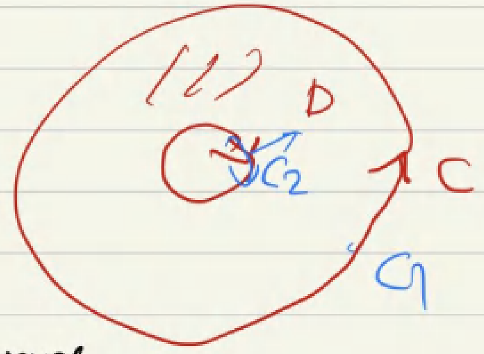
$$= \int_0^{\pi}\int_1^2 \underline{r^2 \sin\theta}\, dr\, d\theta = \int_0^{\pi} \sin\theta\, d\theta \int_1^2 r^2\, dr$$

$$= \frac{14}{3}.$$ ↑ product of

② NOT simply-connected.

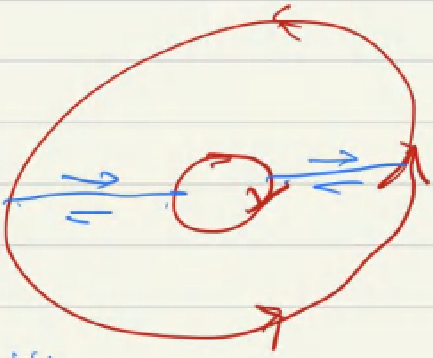
☆ In this case, the boundary ∂D

may have many components. $\partial D = C_1 \cup C_2$



You have to orient these boundary curves. So that D is always on the left as the curve C is traversed.

Then Green's theorem also holds. for these domains:



$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C Pdx + Qdy.$$

$\approx \partial D = \bigcup_i C_i$ with orientation

5. Conservative fields. (on $\mathbb{R}^2$)

Let $\vec{F} = P\vec{i} + Q\vec{j}$ be a vector field. If $\vec{F}$ is a conservative field, we must have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},$$

$\vec{F} = \nabla f.$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

what is the converse?

Theorem. Let $\vec{F} = P\vec{i} + Q\vec{j}$ be a vector field on an open $\boxed{\text{simply connected}}$ region D. Suppose that P and Q

have continuous first-order partial derivatives and

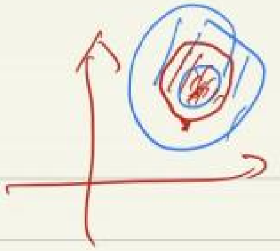
$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{throughout } D.$$

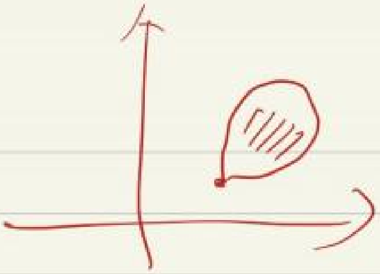
Then $\vec{F}$ is conservative.

conservative

$\Updownarrow$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$





Proof:

If C is any simple closed path in D and R is the region that C encloses, then by Green's theorem,

$$\int_C \vec{F}\cdot d\vec{r} = \int P dx + Q dy$$

$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_R 0 \, dA = 0$$

simply connected, R⊂D.

A curve that is not simple can be broken up into a number of simple curves,

$\int_C \vec{F}\cdot d\vec{r} = 0$ for any closed curve,

$\vec{F}$ is conservative. □

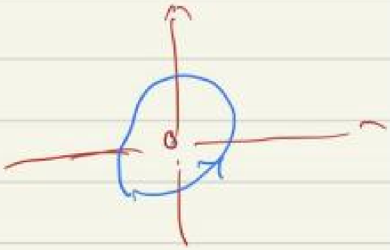
interesting

↑

what if D is NOT simply-connected? No!

Example $\vec{F}(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$, defined on $\mathbb{R}^2 \setminus 0$.

show that $\int_C \vec{F}\cdot d\vec{r} = 2\pi$ for every positively oriented closed path that encloses the origin.



Solution $P = \frac{-y}{x^2+y^2}$, $Q = \frac{x}{x^2+y^2}$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

☆

9,

Now let C be a positively oriented simple closed path that encloses the origin,

Let C' be a small circle
$x^2+y^2=a^2$,
that lies inside C.

(C is compact, so the continuous function $d(O,P)$ has a minimum).
$(P \in C)$

Apply Green's theorem. to D

$$\underline{\int_C} Pdx + Qdy - \underline{\int_{C'}} Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = 0.$$

So we only need to show that

$$\int_C Pdx + Qdy = 2\pi. \longrightarrow$$ but Green's theo— $O \notin$ Disk.

where C: $x^2+y^2=a^2$.

But. $x = a\cos\theta$ $y = a\sin\theta$. $0 \le \theta \le 2\pi$.

$$\vec{F}(\vec{r}(\theta)) \cdot \vec{r}'(\theta) = \frac{(-a\sin\theta)(-a\sin\theta) + (a\cos\theta)(a\cos\theta)}{a^2\cos^2\theta + a^2\sin^2\theta} = 1.$$

So $\underline{\int_C Pdx + Qdy} = \int_0^{2\pi} d\theta = 2\pi$.

$D=$

□

Now $\mathbb{R}^2 \setminus O$ is not simple-connected: the circle $x^2+y^2=1$ encloses the origin O, not in D.

Now $\vec{F}$ satisfies the equation.

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

But the integral of $\vec{F}$ along a closed curve is nonzero.

So $\vec{F}$ is NOT conservative.

Remark: Let $f = \arctan\frac{y}{x} = \theta$,

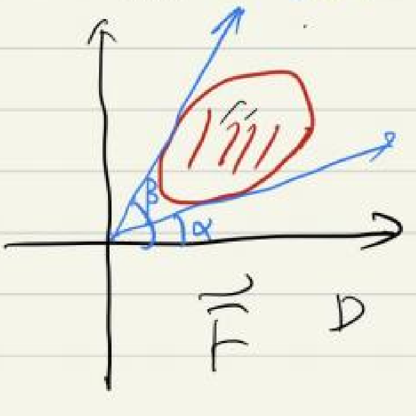
Easy to verify that $\theta(x,y) = $ angle

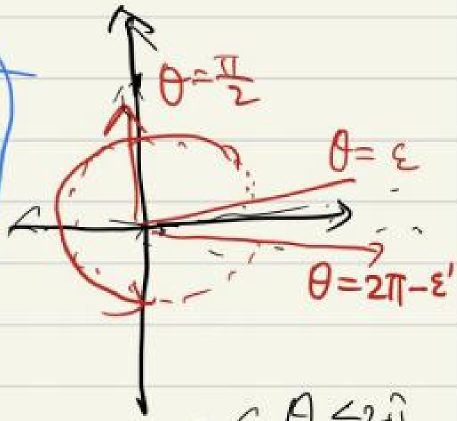
$$\nabla f = \vec{F},$$

☆ But f is NOT a smooth function on $\mathbb{R}^2 \setminus O$.:

To any point (x,y), θ has many branches, and you cannot specify a branch at every point so that θ is continuous.

Of course. if you can choose a branch of θ over some domain D, then $\vec{F}$ is conservative on that domain. $0 \leq \alpha \leq \theta \leq \beta < 2\pi$.





$0 \leq \theta \leq 2\pi$

0?

2π?

6. Vector forms of Green's theorem.

(1) $\vec{F} = P\vec{i} + Q\vec{j}$.

$$\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy.$$

Regarding $\vec{F}$ as a vector field on $\mathbb{R}^3$.

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}.$$

$$\Rightarrow \quad \boxed{\int \vec{F}\cdot d\vec{r} = \iint_D (\operatorname{curl}\vec{F})\cdot\vec{k}\,dA}$$

(2). If C is given by

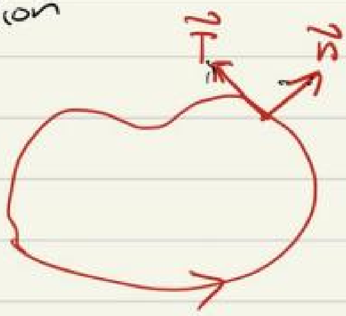
$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} \qquad a \le t \le b, \qquad (\quad)$$

$$\vec{T}(t) = \frac{x'(t)}{|\vec{r}'(t)|}\vec{i} + \frac{y'(t)}{|\vec{r}'(t)|}\vec{j}$$

The outward unit normal vector is

rotation

$$\vec{n} = \frac{y'(t)}{|\vec{r}'(t)|}\vec{i} - \frac{x'(t)}{|\vec{r}'(t)|}\vec{j}$$



$$\int_C \vec{F}\cdot\vec{n}\,ds = \int_a^b \vec{F}\cdot\vec{n}\,|\vec{r}'(t)|\,dt$$

$$= \int_a^b P\,y'(t)dt - Q\,x'(t)dt = \int_C P\,dy - Q\,dx \qquad (-Q, P)$$

Green's

$$= \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA$$

$$\boxed{\int_C \vec{F}\cdot\vec{n}\,ds = \iint_D \operatorname{div}\vec{F}(x,y)\,dA}$$

Homework

P. 3

surface integral. line integral

$\downarrow$ $\downarrow$

Lecture 21.

domains. interval.

Parametric surfaces and surface integrals.

1. How to describe a surface?

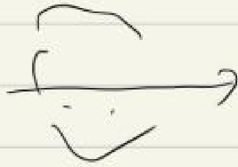
(1) S is the graph of a function $f(x,y)$.

$$z = f(x,y).$$

Example: $z = x^2+y^2$ $z = x^2-y^2$

(2) S is a level surface of a function $F(x,y,z)$.

$$F(x,y,z)=k.$$



Example $x^2+y^2+z^2=1$ ← NOT a graph to (x,y) $z = \pm\sqrt{1-x^2-y^2}$

(3) Parametric surfaces:

describe a surface by a vector function $\vec{r}(u,v)$ of two parameters u and v.: $\vec{r}(u,v): \underbrace{D \to \mathbb{R}^3}$, $\subset \mathbb{R}^2$.

$$\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}.$$

← image of $\vec{r}(u,v)$

The set of all points (x,y,z) in $\mathbb{R}^3$ such that

$x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$

and (u,v) varies throughout D, is called a parametric surface S. and equations.

$x = x(u,v)$, $y = y(u,v)$. $z = z(u,v)$

are called parametric equations of S.

↑ ~~describe~~ represent a point $(x,y,z) \in S$ by (u,v)!

$y = f(x) \to (x, f(x))$

Remark : graph ⟶ level surface

$z = f(x,y)$ $\qquad F(x,y,z) = f(x,y) - z = 0$

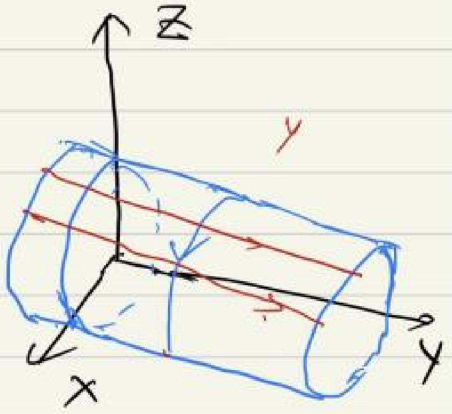
graph ⟶ parametric surface

$z = f(x,y)$ $\qquad \underline{(x,y)} \to (x,y,f(x,y))$. □

Example $\vec{r}(u,v) = 2\cos u\,\vec{i} + v\vec{j} + 2\sin u\,\vec{k}$ $\qquad (u,v) \in \mathbb{R}^2$

$x^2+z^2=4$

(if ix y)



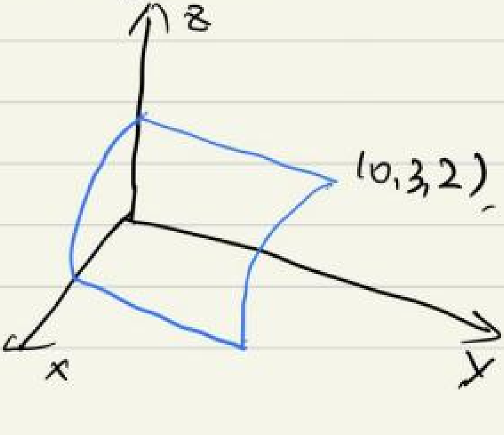
- vertical cross-sections parallel to the xz-plane are all circles with radius 2;
- $y = v$, no restriction is placed on v.

The surface is a circular cylinder.

If we restrict u and v:

$0 \le u \le \frac{\pi}{2}$, $\qquad 0 \le v \le 3$.

we get the quarter-cylinder with length 3.



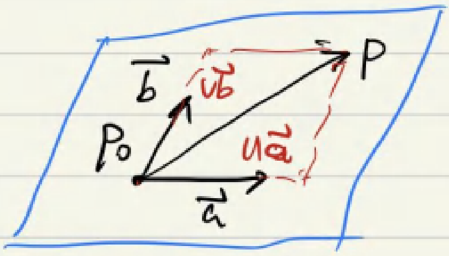
(1). $\vec{a}, \vec{b}, \to \vec{n}$. $\underbrace{(\vec{r}-\vec{r_0}) \cdot \vec{n} = 0}$. $p, \vec{a}, \vec{b}$) ← level surfac

(2) $\vec{r} = \vec{r_0} + u\vec{a} + v\vec{b}$ ← parametric

3.

Example Parametric equations of the plane that passes through the point P_0 with position vector $\vec{r_0}$ and that contains two non-parallel vectors $\vec{a}$ and $\vec{b}$.

If P is any point in the plane, with position vector $\vec{r}$,



There are scalars u and v, such that

$$\vec{r} - \vec{r_0} = \overrightarrow{P_0P} = u\vec{a} + v\vec{b}.$$

So $\vec{r}(u,v) = \vec{r_0} + u\vec{a} + v\vec{b}$.

If $\vec{r} = \langle x, y, z\rangle$, $\vec{r_0} = \langle x_0, y_0, z_0\rangle$, $\vec{a} = \langle a_1, a_2, a_3\rangle$, $\vec{b} = \langle b_1, b_2, b_3\rangle$.

$$x = x_0 + ua_1 + vb_1 \quad y = y_0 + ua_2 + vb_2, \quad z = z_0 + ua_3 + vb_3.$$

Example Find a parametric representation of the sphere

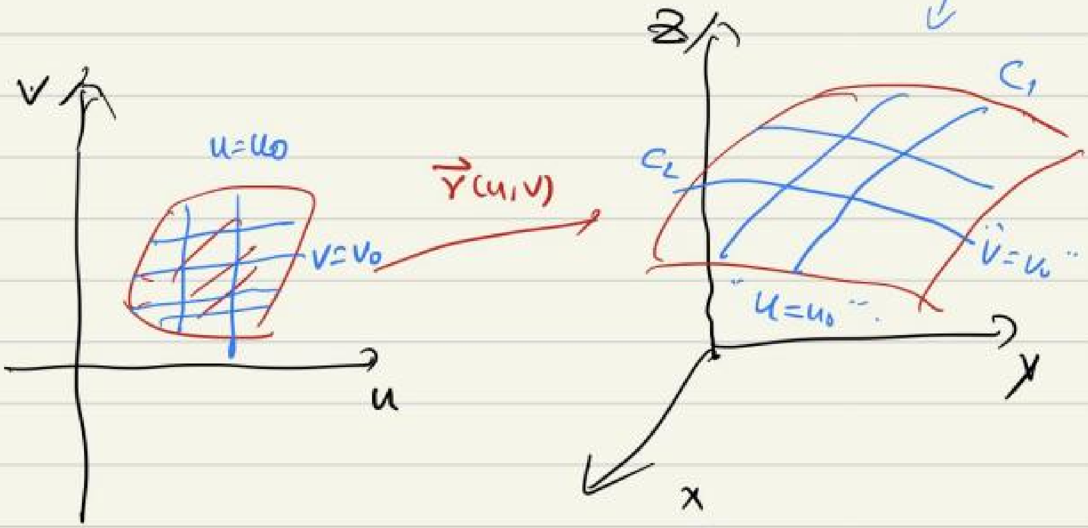
$$x^2 + y^2 + z^2 = a^2.$$

The sphere has a simple representation $\rho = a$ in spherical coordinates, choose the angles ϕ and θ in spherical coordinates as the parameters: ← free variable.

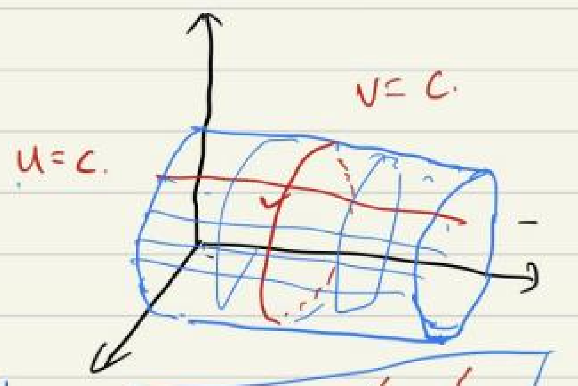
$$x = a\sin\phi\cos\theta, \quad y = a\sin\phi\sin\theta, \quad z = a\cos\phi$$

$$0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi,$$

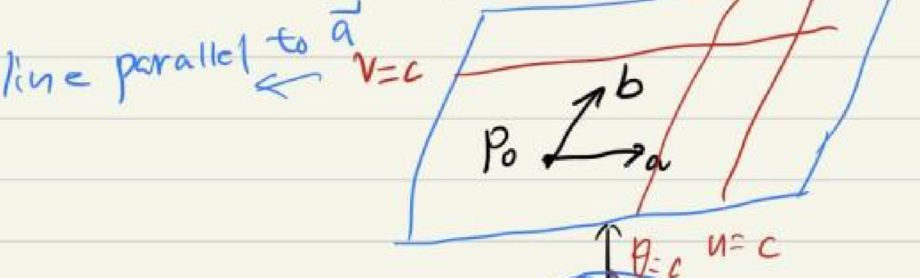
If a parametric surface S is given by a vector function $\vec{r}(u,v)$, then there are two useful families of curves that lie on S, one family with u constant and the other with v constant. These families correspond to vertical and horizontal lines in the uv-plane. We called this curve grid curves.



Example circular cylinder



plane

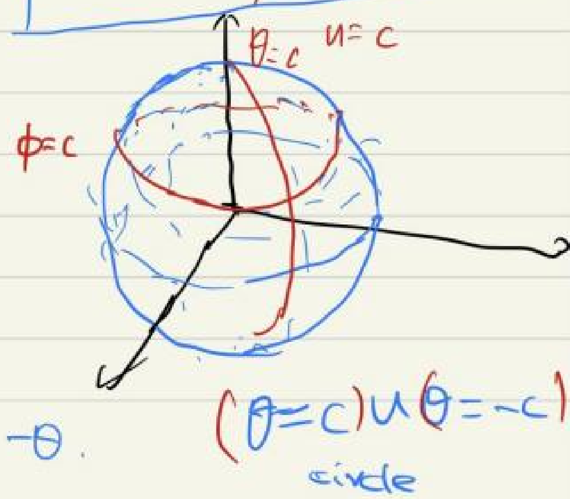


Sphere. $\phi=c$:

circles of constant latitude:

$\theta=c$: meridians

☆ (semi-circles) → $-\theta$.



$\vec{r}(t)$ $\vec{r}'(t)$ smooth

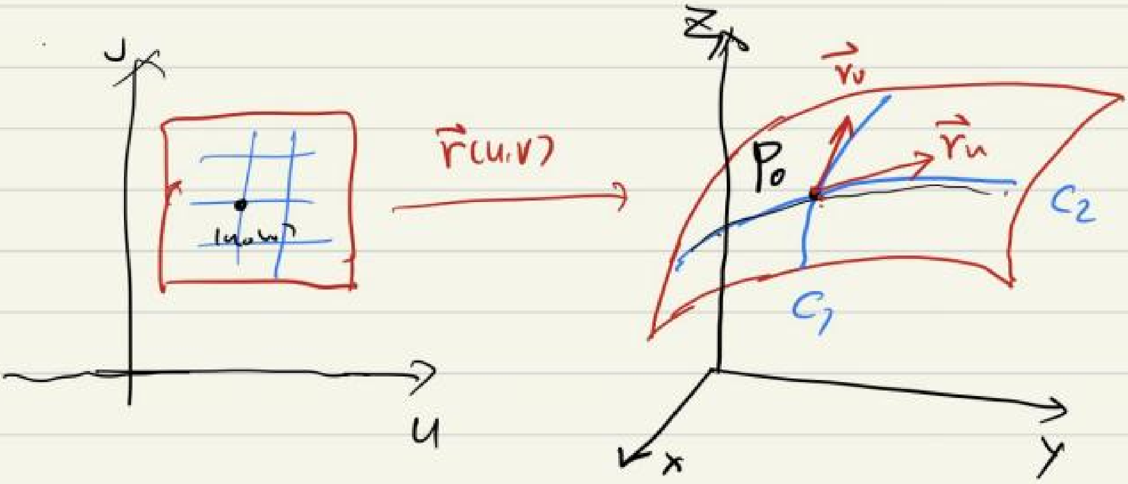
$\vec{r}'(t) = \vec{0}$

cusp

2. Tangent planes.

parametric surface S:

$$\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$$



Let $P_0 = \vec{r}(u_0,v_0)$.

$\vec{\gamma}(u,v_0) = x(u,v_0)\vec{i} + y(u,v_0)\vec{j} + z(u,v_0)\vec{k}$

The grid curve C_2 is the image of the line $v=v_0$,

C_2: $\vec{r}(u,v_0)$. so its tangent vector at P_0. is.

$$\vec{r}_u = \frac{\partial x}{\partial u}(u_0,v_0)\,\vec{i} + \frac{\partial y}{\partial u}(u_0,v_0)\,\vec{j} + \frac{\partial z}{\partial u}(u_0,v_0)\,\vec{k}$$

Similarly, C_1 . . . . $u=u_0$. C_1: $\vec{r}(u_0,v)$.

$$\vec{r}_v = \frac{\partial x}{\partial v}(u_0,v_0)\,\vec{i} + \frac{\partial y}{\partial v}(u_0,v_0)\,\vec{j} + \frac{\partial z}{\partial v}(u_0,v_0)\,\vec{k}$$

1) $\vec{r}_u, \vec{r}_v \neq \vec{0}$

2) NOT parallel ←

If $\vec{r}_u \times \vec{r}_v \neq \vec{0}$, then the surface S is called smooth,

The tangent plane is the plane that contains $\vec{r}_u$ and $\vec{r}_v$.

$\vec{r}_u \times \vec{r}_v$ is a normal vector to the tangent plane.

$\vec{r}(u,v)$ ↘

Example $x=u^2,\ y=v^2,\ z=u+2v.$

$P_0=(1,1,3)=\vec{r}(1,1).$

$\vec{r}_u=2u\vec{i}+\vec{k}$ $\quad\vec{r}_v=2v\vec{j}+2\vec{k}$

$$\vec{r}_u\times\vec{r}_v=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\2u&0&1\\0&2v&2\end{vmatrix}=-2v\vec{i}-4u\vec{j}+4uv\vec{k}$$

normal vector

$\vec{r}_u\times\vec{r}_v(1,1)=\langle-2,-4,4\rangle.$

point

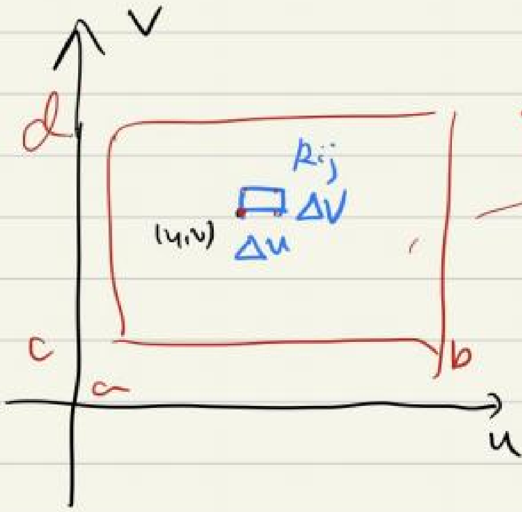
So the tangent plane at $(1,1,3)$ is

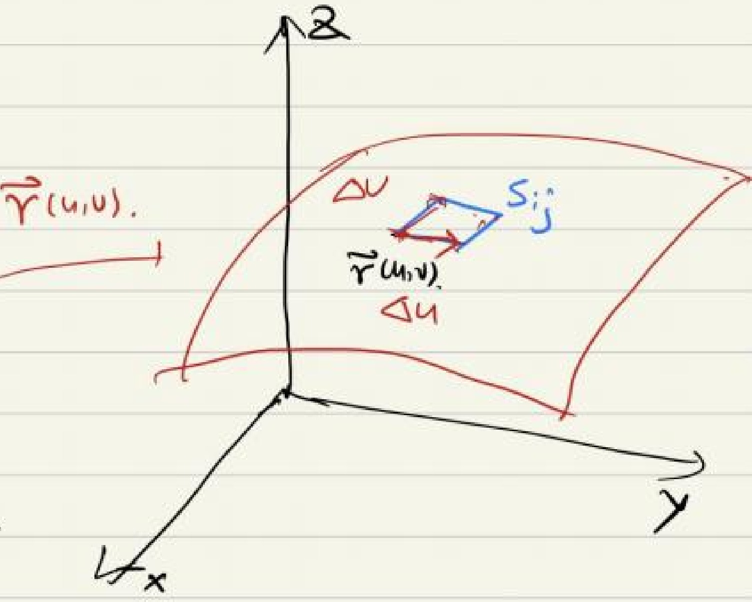
$$-\underline{2}(x-1)-\underline{4}(y-1)+\underline{4}(z-3)=0.$$

$$x+2y-2z+3=0$$

---

3. Surface Area





Let R_{ij} be a small rectangle of area $\Delta u\cdot\Delta v.$

Then

$$\vec{r}(u+\Delta u,v)-\vec{r}(u,v)=\vec{r}_u(u,v)\cdot\Delta u+\text{higher order terms}.$$

$$\vec{r}(u,v+\Delta v)-\vec{r}(u,v)=\vec{r}_v(u,v)\cdot\Delta v+\text{higher order terms},$$

$$R_{ij} \longrightarrow S_{ij} = |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$$

7,

Let S_{ij} be the image of R_{ij}.

Then the area of S_{ij} (surface) (nonlinear)

$\approx$ the area of the parallelgram spanned by $\vec{r}(u+\Delta u) - \vec{r}(u,v)$, and $\vec{r}(u,v+\Delta v) - \vec{r}(u,v)$ (linear)

$$= \left|(\vec{r}_u(u,v)\Delta u) \times (\vec{r}_v(u,v)\Delta v)\right|$$

$$= \left|\vec{r}_u(u,v) \times \vec{r}_v(u,v)\right| \Delta u \Delta v. \quad + \text{h.o.t}$$

Let $\Delta u, \Delta v \to 0$, we get $\boxed{\text{Area}(S) = \lim_{m,n\to\infty} \sum_m \sum_n |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v}$

If a smooth parametric surface S is given by the equatio

$$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}. \qquad (u,v) \in D,$$

and S is ☆covered just once as (u,v) ranges throughout the parameter domain D, then the surface area of S is

$$A(S) = \iint_D |\underline{\vec{r}_u \times \vec{r}_v}| \, dA.$$

where $\vec{r}_u = \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j} + \frac{\partial z}{\partial u}\vec{k}$,

$$\vec{r}_v = \frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j} + \frac{\partial z}{\partial v}\vec{k}.$$

Find area: $\begin{cases} 1)\ \text{parametrization}\quad \vec{r}(u,v)\quad D. \\ 2)\ \iint_S \longrightarrow \iint_D. \\ \quad dA \longrightarrow |\vec{r_u}\times\vec{r_v}|\,dA \end{cases}$

Example Find the surface area of a sphere of radius a.

Solution: parametric representation.

$$x = a\sin\phi\cos\theta, \qquad y = a\sin\phi\sin\theta \qquad z = a\cos\phi$$

$$D = \{(\phi,\theta) \mid 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi\}.$$

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= a^2\sin^2\phi\cos\theta\,\vec{i} + a^2\sin^2\phi\sin\theta\,\vec{j} + a^2\sin\phi\cos\phi\,\vec{k}$$

$$|\vec{r}_\phi \times \vec{r}_\theta| = a^2\sin\phi \geq 0.$$

$$A(S) = \iint_D a^2\sin\phi\, dA = \int_0^{2\pi}\int_0^{\pi} a^2\sin\phi\, d\phi\, d\theta \leftarrow \text{product.}$$

$$= a^2\int_0^{2\pi} d\theta \int_0^{\pi} \sin\theta\, d\theta = a^2 \times 2\pi \times 2 = 4\pi a^2.$$

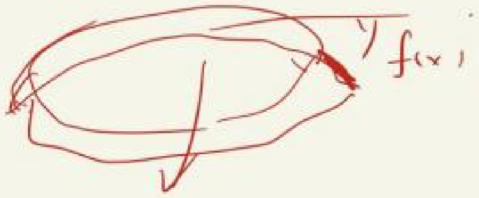
Example Area surface of a graph: $z = f(x,y)$

parametric equation:

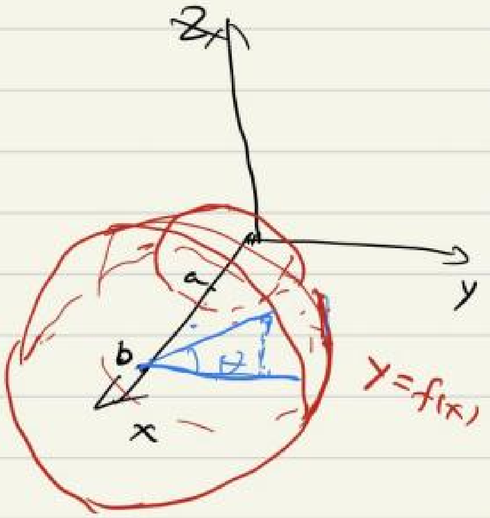
Solution: $x = x \qquad y = y \qquad z = f(x,y). \qquad (x,y) \in D \rightarrow$ projection of S xy-plane

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = -f_x\vec{i} - f_y\vec{j} + k.$$

$$A(S) = \iint_D \sqrt{1 + (f_x)^2 + (f_y)^2}\, dx\,dy$$



Example Surface of Revolution.



Let's consider the surface S obtained by rotating the curve $y=f(x)$, $a \le x \le b$. about the x-axis. where $f(x) \ge 0$.

Let θ be the angle of rotation.

Parametric equations:

$$\begin{cases} x=x \quad y=f(x)\cos\theta, \quad z=f(x)\sin\theta \\ a \le x \le b \ , \quad 0 \le \theta \le 2\pi. \end{cases}$$

$\vec{r}(u,v)$

D

$$\vec{r}_x = \vec{i} + f'(x)\cos\theta\vec{j} + f'(x)\sin\theta\vec{k}$$

$$\vec{r}_\theta = -f(x)\sin\theta\vec{j} + f(x)\cos\theta\vec{k}$$

$$\vec{r}_x \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & f'(x)\cos\theta & f'(x)\sin\theta \\ 0 & -f(x)\sin\theta & f(x)\cos\theta \end{vmatrix}$$

$$= f(x)f'(x)\vec{i} - f(x)\cos\theta\vec{j} - f(x)\sin\theta\vec{k}$$

$$|\vec{r}_x \times \vec{r}_\theta| = f(x)\sqrt{1+(f'(x))^2}. \Leftarrow$$

$f(x) \ge 0$.

$|f(x)|$ otherwise

$$A = \iint_D |\vec{r}_x \times \vec{r}_\theta| \, dA$$

$$= \int_0^{2\pi}\int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx\,d\theta$$

$$= \underline{2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx} \Leftarrow$$

Calculus 1.

4. Surface integrals (of functions).

$C: \ \vec{r}: [a,b] \to \mathbb{R}^3.$

length

$$L(C) = \int_C \underline{ds} = \int_a^b |\vec{r}'(t)|dt$$

$f(x_i^*)\underline{\underline{\Delta s_i}}$

function $f(x,y,z)$

$$\int_C f ds = \int_a^b f(\underline{x(t), y(t), z(t)}) |\vec{r}'(t)| dt$$

Suppose that a surface S has a vector equation

$$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k} \qquad (u,v) \in D$$

We first assume that the parameter domain D is a rectangle and we divide it into subrectangles R_{ij}. Then the surface S is divided into corresponding patches S_{ij}.

We evaluate f at a point $P_{ij}^* \leftarrow S_{ij}$ in each patch, multiply by the area ΔS_{ij} of the patch, and form the Riemann sum.

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\underline{\underline{\Delta S_{ij}}}$$ area

~~ΔS_i~~

Then we take the limit as the number of patches increases and define the <u>surface integral of f over the surface S.</u>

definition $\rightarrow$

$$\boxed{\iint_S f(x,y,z)\,dS = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij}}$$

$$\Delta S_{ij} = |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$$

$\longrightarrow$ $\underline{D} \subseteq \underline{\mathbb{R}^2}$ linear flat domain

We know that $\Delta S_{ij} \approx |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$, so.

computation →

$$\boxed{\iint_{\underline{S}} f(x,y,z)\, dS = \iint_{\underline{\underline{D}}} f(\vec{r}(u,v))\, |\vec{r}_u \times \vec{r}_v|\, dA.}$$

← double integral

Example $\iint_S x^2 dS$, $S: x^2+y^2+z^2=1$.

Solution. $x=\sin\phi\cos\theta$, $y=\sin\phi\sin\theta$, $z=\cos\phi$.

$0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$,

$|\vec{r}_\phi \times \vec{r}_\theta| = \sin\phi$.

$$\iint_S x^2 dS = \iint_D \sin^2\phi \cos^2\theta \cdot \sin\phi \, dA$$

← product, over rectangle.

$$= \int_0^{2\pi} \cos^2\theta \, d\theta \cdot \int_0^{\pi} \sin^3\phi \, d\phi.$$

$$\int_0^{2\pi} \cos^2\theta \, d\theta = \int_0^{2\pi} \frac{1+\cos 2\theta}{2} \, d\theta = \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]\Big|_0^{2\pi} = \pi$$

$$\int_0^{\pi} \sin^3\phi \, d\phi = \int_0^{\pi} \underset{\sim}{\sin\phi} - \underline{\sin\phi\cos^2\phi \, d\phi}$$

$d(\cos\phi) = -\sin\phi \, d\phi$.

$$= -\cos\phi + \frac{1}{3}\cos^3\phi \Big|_0^{\pi} = 2 - \frac{2}{3} = \frac{4}{3}$$

$$\iint_S x^2 dS = \frac{4}{3}\pi.$$

$\iint_S \longrightarrow \iint_D$.

$f(x,y,z) \longrightarrow f(\vec{r}(u,v))$

$dS \longrightarrow |\vec{r}_u \times \vec{r}_v| \, dA$

integral/ interval ← line integral $ds = |\vec{r}'(t)|dt$ $\int ds$ $\int f ds$ ← ⌝ surface integral. $dS = |\vec{r}_u \times \vec{r}_v| dA$ $\int dA$ $\int f dA$ 12,

↓ double integral / region $D \subseteq \mathbb{R}^2$

If S is the graph $z = g(x,y)$.

$$dS = \sqrt{1+(g_x)^2+(g_y)^2}\, dx dy$$

$$\boxed{\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\, dS.}$$

Example $\iint_S y\,dS$: $S: z = \underline{x+y^2}$. $0 \le x \le 1$. $0 \le y \le 2$.

↖ graph.

Solution: $\frac{\partial z}{\partial x} = 1$ $\frac{\partial z}{\partial y} = 2y.$

$$\iint_S y\,dS = \iint_D y\sqrt{2+4y^2}\,dA$$

$$= \sqrt{2}\int_0^1 dx \int_0^2 y\sqrt{1+2y^2}\,dy$$

$$= \sqrt{2}\left(\tfrac{1}{4}\right)\cdot\tfrac{2}{3}(1+2y^2)^{3/2}\Big|_0^2 = \frac{13\sqrt{2}}{3},$$

If S is a piecewise-smooth surface, that is, a finite union of smooth surfaces $S_1 \cdots S_n$, that intersect only along their boundaries, then the surface integral of f over S is defined by

$$\iint_S f(x,y,z)\,dS = \sum_{i=1}^{n} \iint_{\underline{\underline{S_i}}} f(x,y,z)\,dS.$$

↑ smooth surfaces.

Lecture 22.

1. Oriented surfaces

$\vec{F}\cdot d\vec{r}$ $\quad \vec{r}(t)$

- $\int_C P dx + Q dy + R dz$ relies on the orientation of C.

Any curve is orientable.

- surface integrals of vector fields, also rely on the orientation of the surface S.

- Surface integrals of vector fields is only defined for orientable surfaces.

$\vec{r}(u,v)$ $\quad \vec{r}_u\times\vec{r}_v \neq \vec{0}$

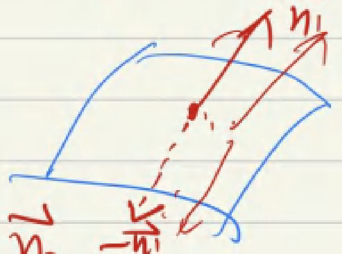
Let S be a surface that has a tangent plane at every point (x,y,z) on S, (except at any boundary point). There are two unit normal vectors $\vec{n}_1$ and $\vec{n}_2 = -\vec{n}_1$ (at any point (local).

S

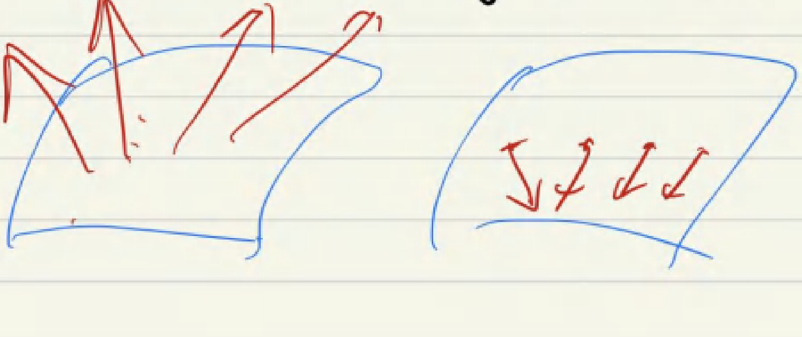
$\vec{n}_1 = \frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}$,

$\vec{n}_2 = -\vec{n}_1$

If it is possible to choose a unit normal vector $\vec{n}$ at every point $\vec{n}$ so that $\vec{n}$ varies continuously over S, then S is called an oriented surface, and the given choice of $\vec{n}$. provides S with an orientation. ✓
(vector fields)



There are two possible orientations for any orientable surface.



Example $S: z = g(x, y)$.

$\vec{r}_x = (1, 0\ g_x)$. $\vec{r}_y = (0, 1, g_y)$.

$\vec{r}_x \times \vec{r}_y = \langle -g_x, -g_y, 1\rangle$,

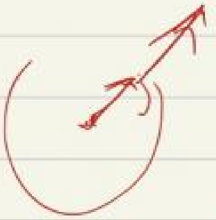
$$✓\ \vec{n} = \frac{(\vec{r}_x \times \vec{r}_y|}{|\vec{r}_x \times \vec{r}_y|} = \frac{-\frac{\partial g}{\partial x}\vec{i} - \frac{\partial g}{\partial y}\vec{j} + \vec{k}}{\sqrt{1 + (\frac{\partial g}{\partial x})^2 + (\frac{\partial g}{\partial y})^2}}$$

provides an orientation. <upward orientation>.

Example Level surfaces:

$S: F(x, y, z) = k$, $\nabla F|_S \neq \vec{0}$ (smooth).

then S is orientable; $\vec{n} = \frac{\nabla F}{|\nabla F|}$.



$x^2 + y^2 + z^2 = 1$. $\nabla F = (2x, 2y, 2z)$.

$\vec{n} = (x, y, z)$

normal vector <=> posit
vect

If a parametric surface S is orientable, $\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$

provides an orientation. $\vec{r}(u,v)$

Vector field

$\vec{n_2} = \frac{\vec{r}_v \times \vec{r}_u}{|\vec{r}_v \times \vec{r}_u|}$

Example Sphere: $x^2 + y^2 + z^2 = a^2$.

$$\vec{r}(\phi, \theta) = a\sin\phi\cos\theta\,\vec{i} + a\sin\phi\sin\theta\,\vec{j} + a\cos\phi\,\vec{k}$$

$$\vec{r}_\phi \times \vec{r}_\theta = a^2\sin^2\phi\cos\theta\,\vec{i} + a^2\sin^2\phi\sin\theta\,\vec{j} + a^2\sin\phi\cos\phi\,\vec{k}$$

(ϕ, θ) ✓ (θ, ϕ)

(1), $\vec{n}$ // position.
$\vec{n} = \vec{r}/a$ $\searrow$ a

$|\vec{r}_\phi \times \vec{r}_\theta| = a^2\sin\phi$ (2). (ϕ,θ). → give the correct orientation.

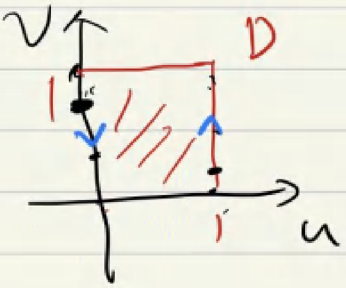
$$\Rightarrow \underline{\vec{n}} = \sin\phi\cos\theta\,\vec{i} + \sin\phi\sin\theta\,\vec{j} + \cos\phi\,\vec{k} = \underline{\tfrac{1}{a}\vec{r}(\phi,\theta)}$$

☆ For a closed surface, that is, a surface that is the boundary of a solid region E, the convention is that the positive orientation is the one for which the normal vectors point outward from E.

closed surface: spheres.
non-closed: plane, graph

Example Nonorientable surface: the Möbius strip. M

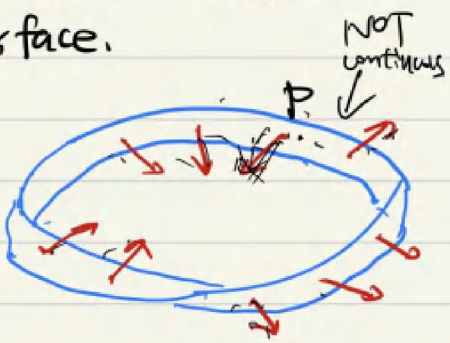
Let D be $[0,1]\times[0,1]$.

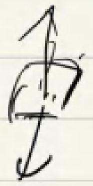


Topologically, the Möbius strip is a quotient of D; we identify the points. $(0,t)$ and $(1,1-t)$.

We cannot do this in $\mathbb{R}^2$, but we can embed the Möbius strip in $\mathbb{R}^3$. So M is a parametric surface.

But easy to see that M is NOT orientable.





Nonorientable: there is one face
orientable: 2 faces

Surface integral of vector field
$\downarrow$ surface function
$\downarrow$ double integral over D.

4.

2. Surface Integrals of Vector Fields

$= P\vec{i} + Q\vec{j} + R\vec{k}$

Definition. If $\vec{F}$ is a continuous vector field, defined on an oriented surface S, with unit normal vector $\vec{n}$, then the surface integral of $\vec{F}$ over S is

$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\, dS}$$

↑ orientable

$\vec{F}\cdot\vec{n}$ function on S

The integral is called the flux of $\vec{F}$ across S.

If S is given by a vector function $\vec{r}(u,v)$, then

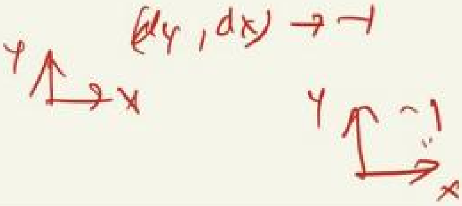
$$\vec{n} = \frac{\vec{r_u}\times\vec{r_v}}{|\vec{r_u}\times\vec{r_v}|}. \leftarrow \text{natural orientation of parametric surfaces}$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \underline{\vec{F}\cdot\frac{\vec{r_u}\times\vec{r_v}}{|\vec{r_u}\times\vec{r_v}|}}\, dS \qquad x,y,z \to (u,v)$$

$$= \iint_D \left[\vec{F}(\vec{r}(u,v))\cdot\frac{\vec{r_u}\times\vec{r_v}}{|\vec{r_u}\times\vec{r_v}|}\right] \underline{|\vec{r_u}\times\vec{r_v}|\, dA}$$

where D is the parameter domain. Thus we have

$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot(\vec{r_u}\times\vec{r_v})\, dA = \iint_D \begin{vmatrix} P & Q & R \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} du\,dv.}$$



(dy, dx) → -1

(dx, dy) → 1

"orientation"

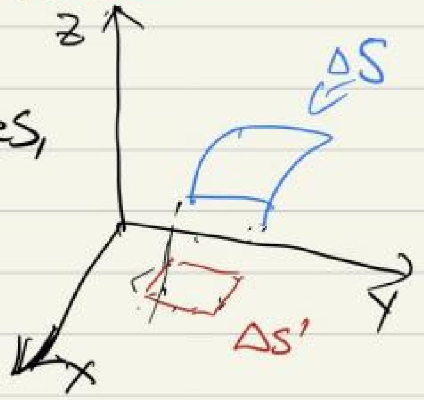
signed area

Remark

$$\iint_S \vec{F}\cdot d\vec{S}$$

$$= \iint_D \left[P\,\frac{\partial(y,z)}{\partial(u,v)} + Q\,\frac{\partial(z,x)}{\partial(u,v)} + R\,\boxed{\frac{\partial(x,y)}{\partial(u,v)}} \right] du\,dv.$$

☆



Let ΔS be a small patch of the surfaces,

Consider its projection to the xy-plane:

ΔS': represented by

surface

$(x(u,v),\ y(u,v),\ 0)$,

so the area is

$$\left\| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} \right\| \Delta u \Delta v = \underbrace{\left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v}_{\geq 0}.$$

However, we have to consider the orientation, so we use the symbol dxdy instead of dA, to indicate an orientation.

signed area.

$$dxdy = \frac{\partial(x,y)}{\partial(u,v)}\,du\,dv.$$

$\iint_D dxdy$

dA

NOT the same

We also have dydz, dzdx.

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S P\,dydz + Q\,dzdx + R\,dxdy.$$

$$\iint_S \vec{F}\cdot\vec{n}\, dS$$

$$= \underline{\iint_D F\cdot(\vec{r_u}\times\vec{r_v})\, dA} \quad 6.$$

Example $\iint_S \vec{F}\cdot d\vec{S}$.

$\vec{F} = z\vec{i} + y\vec{j} + x\vec{k}$.

$S: x^2+y^2+z^2=1$.

$\iint_S \longrightarrow \iint_D$.

$\iint \vec{F}\cdot\vec{n} \to F\cdot(\vec{r_u}\times\vec{r_v})$

$d\vec{S} \longrightarrow dA$

Solution: $\underline{\vec{r}(\phi,\theta)} = \sin\phi\cos\theta\,\vec{i} + \sin\phi\sin\theta\,\vec{j} + \cos\phi\,\vec{k}$

$0\le\phi\le\pi, \quad 0\le\theta\le 2\pi$.

$\vec{F}(\vec{r}(\phi,\theta)) = \underline{\cos\phi}\,\vec{i} + \sin\phi\sin\theta\,\vec{j} + \sin\phi\cos\theta\,\vec{k}$

$\vec{r_\phi}\times\vec{r_\theta} = \vec{r}(\phi,\theta)$. $\leftarrow$ $a=1$

$\vec{F}(\vec{r}(\phi,\theta))\cdot\vec{r}(\phi,\theta) = \vec{F}\cdot(\vec{r_\phi}\times\vec{r_\theta})$

$= 2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta$.

$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot(\vec{r_\phi}\times\vec{r_\theta})\, dA$

produces

$= \int_0^{2\pi}\int_0^{\pi}(\underline{2\sin^2\phi\cos\phi}\cos\theta + \sin^3\phi\sin^2\theta)\, d\phi\, d\theta$

$= 2\int_0^{\pi}\sin^2\phi\cos\phi\, d\phi\int_0^{2\pi}\underline{\cos\theta\, d\theta} + \int_0^{\pi}\sin^3\phi\, d\phi\int_0^{2\pi}\sin^2\theta\, d\theta$

$\downarrow$ 0

$= \int_0^{\pi}\sin^3\phi\, d\phi\int_0^{2\pi}\sin^2\theta\, d\theta = \frac{4}{3}\pi$.

(Lecture note 21)

graphs

S: parametric surface (x, y)) $\vec{r}_x \times \vec{r}_y \rightarrow$ upward orientation

7,

If S: $z = g(x, y)$, $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

$$\vec{F} \cdot (\vec{r}_x \times \vec{r}_y) = (P\vec{i} + Q\vec{j} + R\vec{k})\left(-\frac{\partial g}{\partial x}\vec{i} - \frac{\partial g}{\partial y}\vec{j} + \vec{k}\right)$$

$=$

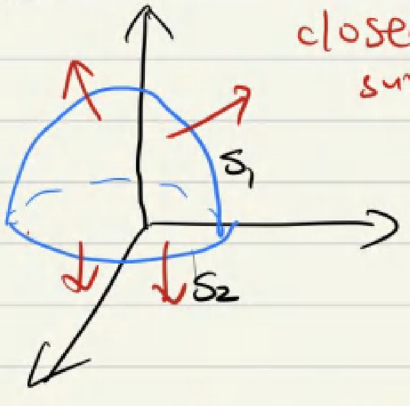
☆ $$\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left(\underline{-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y}} + R\right) dA}$$

dA
for the xy-pla

☆ If S is downward oriented, multiply by −1.

Example $\underline{\iint_S \vec{F} \cdot d\vec{S}}$ $\vec{F}(x, y, z) = y\vec{i} + x\vec{j} + z\vec{k}$.

S: the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$



closed surface

Solution Let S_1: $z = 1 - x^2 - y^2$, upward oriented.

S_2: $z = 0$, downward oriented.

S_1: $g = 1 - x^2 - y^2$.

$$-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R$$

$\leftarrow P = y.\ Q = x \quad R = z = 1 - x^2 - y^2$

$= 1 + 4xy - x^2 - y^2$.

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_D (1 + 4xy - x^2 - y^2)\, dA$$

polar coordinates
1) D simple
2) function

$$= \int_0^{2\pi}\int_0^1 (1 + 4r^2\cos\theta\sin\theta - r^2)\, \underline{r}\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 (r - r^3 + 4r^3\cos\theta\sin\theta)\, dr\, d\theta$$

$$= \int_0^{2\pi}\left(\frac{1}{4} + \cos\theta\sin\theta\right)d\theta = \frac{1}{4}\cdot 2\pi + 0 = \frac{\pi}{2}.$$

$$\leftarrow -\iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$

$g = 0.$

S_2: $\vec{n} = -k.$

$\vec{F}\cdot\vec{n} = \vec{F}\cdot(-\vec{k}) = 0.$ $-\iint 0\, dA$

$\iint_{S_2} \vec{F}\cdot d\vec{S} = 0$

$\iint_S \vec{F}\cdot d\vec{S} = \frac{\pi}{2}.$

□

3. Stokes' Theorem.

Green's Theorem

generalization

double integral over a plane region D. $\subset \mathbb{R}^2$ ⟺ line integral around its plane boundary curve $C = \partial D$

Stokes' Theorem.

surface integral over a surface S $\subset \mathbb{R}^3$ ⟺ line integral around the boundary curve of S

Let S be an oriented surface with unit normal vector $\vec{n}$. The orientation of S induces the positive orientation of the boundary curve C:

If you walk in the positive direction around C with your head pointing in the direction of $\vec{n}$, then the surface will always be on your left.

Stokes' Theorem

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth curve C with positive orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

induced by S

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}.$$

Remark If S is flat and lies in the xy-plane, with upward orientation, the unit normal vector is $\vec{k}$. Stokes' theorem becomes

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_S (\operatorname{curl}\vec{F})\cdot\vec{k}\, dA.$$

Green's theorem is a special case of Stokes' theorem.

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, shift left] & S^3 \arrow[l, dashed, shift left, "\text{pulled back}"] \end{tikzcd}$$

Proof: Let $S: \underset{\mathbb{R}^2}{\underset{\cap}{D}} \longrightarrow \mathbb{R}^3$. $\vec{r}(u,v)$.

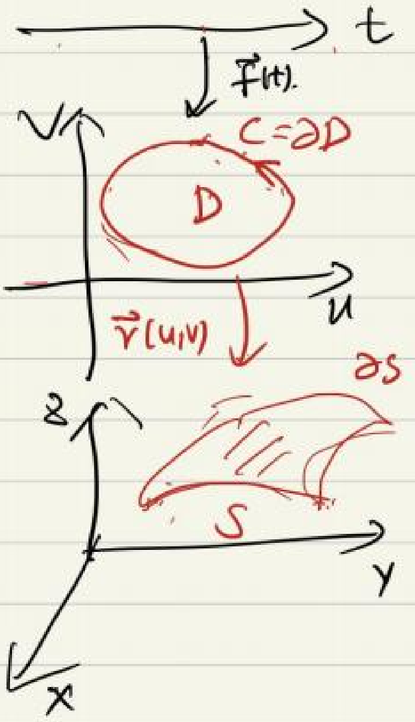
With this parametrization, everything can be "pulled-back" to a region D in $\mathbb{R}^2$. So we may expect to prove Stokes' theorem by Green's theorem.

Let $\vec{f}(t): \mathbb{R}\to\mathbb{R}^2$ be a parametrization of ∂D, with positive orientation.

$$\vec{f}(t) = (u(t), v(t)),\ \alpha\le t\le\beta,$$

Then, C

$$\vec{\gamma} = \vec{r} \circ \vec{f}(t), \quad \alpha \le t \le \beta.$$



line integr in $\mathbb{R}^2$

Consider $\int_C P\,dx$.

← Chain rule.

$$\frac{dx}{dt} = \frac{\partial x}{\partial u}u'(t) + \frac{\partial x}{\partial v}v'(t).$$

$t \to (u,v) \to x$

$$\Rightarrow \int_C P\,dx$$

dx

$$= \int_\alpha^\beta P \circ \vec{r} \circ \vec{f}(t)\left(\frac{\partial x}{\partial u}u'(t) + \frac{\partial x}{\partial v}v'(t)\right)dt$$

line integral in the uv plane.

$$= \int_{\partial D} P\circ\vec{r}\,\frac{\partial x}{\partial u}\,du + P\circ\vec{r}\,\frac{\partial x}{\partial v}\,dv.$$

Apply Green's theorem to this line integral: (region in $\mathbb{R}^2$)

$$\frac{\partial}{\partial u}\left(P\circ\vec{r}\,\frac{\partial x}{\partial v}\right) = \left(\frac{\partial P}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial P}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial u}\right]\frac{\partial x}{\partial v} + P\circ\vec{r}\cdot\frac{\partial^2 x}{\partial u\partial v}.$$

Leibniz rule + chainrule

$$\frac{\partial}{\partial v}\left(P\circ\vec{r}\,\frac{\partial x}{\partial u}\right] = \left(\frac{\partial P}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial P}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial v}\right)\frac{\partial x}{\partial u} + P\circ\vec{r}\,\frac{\partial^2 x}{\partial u\partial v}.$$

$$\Rightarrow \frac{\partial}{\partial u}\left(P\circ\vec{r}\,\frac{\partial x}{\partial v}\right) - \frac{\partial}{\partial v}\left(P\circ\vec{r}\,\frac{\partial x}{\partial u}\right)$$

$$= \frac{\partial P}{\partial z}\circ\vec{r}\cdot\frac{\partial(z,x)}{\partial(u,v)} - \frac{\partial P}{\partial y}\circ\vec{r}\,\frac{\partial(x,y)}{\partial(u,v)}.$$

$$\int_C P\,dx = \iint_D \Big(\frac{\partial P}{\partial z} \circ \vec{r} \, \frac{\partial(z,x)}{\partial(u,v)} - \frac{\partial P}{\partial y} \circ \vec{r} \, \frac{\partial(x,y)}{\partial(u,v)} \Big) du\,dv.$$

Similarly.

$$\int_C Q\,dy = \iint_D \frac{\partial Q}{\partial x} \circ \vec{r} \, \frac{\partial(x,y)}{\partial(u,v)} - \frac{\partial Q}{\partial z} \circ \vec{r} \, \frac{\partial(y,z)}{\partial(u,v)} \Big) du\,dv.$$

$$\int_C R\,dz = \iint_D \frac{\partial R}{\partial y} \circ \vec{r} \, \frac{\partial(y,z)}{\partial(u,v)} - \frac{\partial R}{\partial x} \, \frac{\partial(z,x)}{\partial(u,v)} \Big) du\,dv.$$

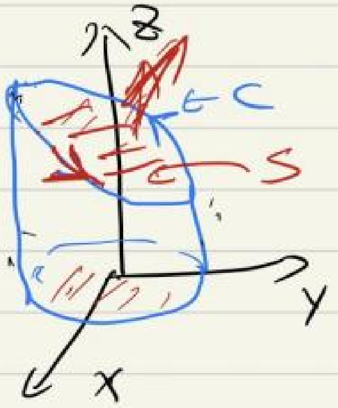
Then take the sum of the three equations □

(curl $\vec{F}$ simple)

Example $\int_C \vec{F} \cdot d\vec{r}$. $\vec{F}(x,y,z) = -y^2 \vec{i} + x\vec{j} + z^2 \vec{k}$. (C: orientation is given)

C: intersection of : $y+z=2$, $x^2+y^2=1$

Solution $\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = \boxed{(1+2y)\vec{k}}$



Orient S upward.

S is a graph : $z = 2-y$ over. D: $x^2+y^2 \le 1$.

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \underline{\iint_D (1+2y)\,dA}$$

$$= \int_0^{2\pi} \int_0^1 (1 + 2r\sin\theta)\, r\,dr\,d\theta.$$

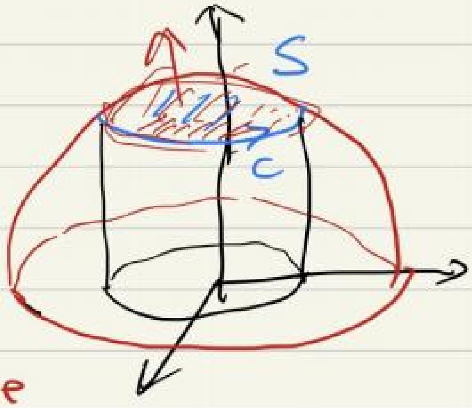
$= \pi.$

$$\boxed{-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R}$$

$\| \; 0$ $\qquad (1+2y)$

Example $\iint_S \text{curl}\, \vec{F} \cdot d\vec{S}$.

$\vec{F} = xz\,\vec{i} + yz\,\vec{j} + xy\,\vec{k}$.



S: the part of the sphere $x^2+y^2+z^2=4$, that lies inside the cylinder $x^2+y^2=1$ and above the xy-plane. $z \geq 0$.

← C simple

Solution 1. C: $x^2+y^2=1$, $z=\sqrt{3}$

$\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + \sqrt{3}\,\vec{k} \quad 0 \leq t \leq 2\pi.$

$\vec{r}\,'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j}$

$\vec{F}(\vec{r}(t)) = \underset{z}{\sqrt{3}}\,\underset{x}{\cos t}\,\vec{i} + \underset{z}{\sqrt{3}}\,\underset{y}{\sin t}\,\vec{j} + \underset{x}{\cos t}\,\underset{y}{\sin t}\,\vec{k}.$

By Stokes' theorem,

$$\iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= \int_0^{2\pi} (-\sqrt{3}\cos t \sin t + \sqrt{3}\sin t \cos t)\,dt = 0.$$

13.

← change area

Solution.
$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & yz & xy \end{vmatrix}$$

$$= (x-y)\vec{i} + (x-y)\vec{j}$$

Let $S' = \{x^2+y^2 \le 1, \underline{z=3}\}$. a disk

$$\vec{n} = \vec{k}.$$

By Stokes' theorem,

$$\iint_{\underline{\underline{S}}} \operatorname{curl}\vec{F}\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r} = \iint_{\underline{\underline{S'}}} \operatorname{curl}\vec{F}\cdot d\vec{S'}$$

But now. $\underline{\operatorname{curl}\vec{F}\cdot\vec{n} = 0}$.

$$\Rightarrow \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = 0. \quad \square$$

$\operatorname{curl}\vec{F} = (x-y, x-y, 0)$.

$\vec{n} = (0, 0, 1)$

C is the common boundary of S and S'

$$\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_{S'} \operatorname{curl}\vec{F}\cdot d\vec{S}. \quad \square$$

Lecture 23.

1. The divergence theorem.

Green's theorem

$$\int_C \vec{F}\cdot\vec{n}\,ds = \iint_D \operatorname{div}\vec{F}(x,y)\,dA.$$

dim 2 → dim 3

$$\int \vec{F}\cdot d\vec{r} = \iint \operatorname{curl}\vec{F}\cdot\vec{k}\,dA$$

$D \subseteq \mathbb{R}^2$

↓

surface

Let E be a region in $\mathbb{R}^3$. E is called a simple solid region if it is simultaneously of types 1, 2, and 3. ← sphere

The divergence Theorem

Let E be a simple solid region and let S be the boundary surface of E, given the positive (outward) orientation. Let $\vec{F}$ be a vector field whose component functions have continuous partial derivatives on some open region that contains E. Then

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\,dV$$

$$\operatorname{div}\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Proof: Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$.

$$\iiint_E \operatorname{div}\vec{F}\,dV = \iiint_E \frac{\partial P}{\partial x}\,dV + \iiint_E \frac{\partial Q}{\partial y}\,dV + \iiint_E \frac{\partial R}{\partial z}\,dV.$$

If $\vec{n}$ is the unit outward normal of S,

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \underbrace{\vec{F}\cdot\vec{n}\,dS}$$

$$= \iint_S P\vec{i}\cdot\vec{n}\,dS + \iint_S Q\vec{j}\cdot\vec{n}\,dS + \iint_S R\vec{k}\cdot\vec{n}\,dS$$

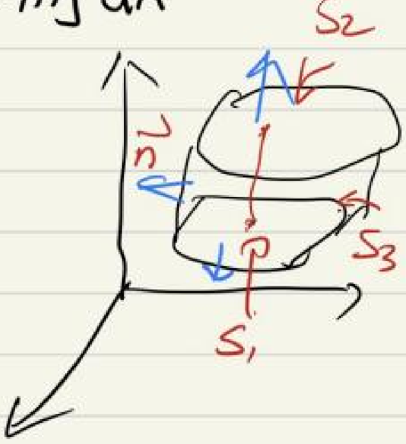
We prove that

$$\iint_S R\vec{k}\cdot\vec{n}\,dS = \iiint_E \frac{\partial R}{\partial z}\,dV.$$

Write $E = \{(x,y,z) \mid (x,y)\in D,\ u_1(x,y)\le z \le u_2(x,y)\}$. type I.

$$\iiint_E \frac{\partial R}{\partial z}\,dV = \iint_D \Big[{}_{u_1(x,y)}^{u_2(x,y)} \frac{\partial R}{\partial z}(x,y,z)\,dz\Big]\,dA \qquad \text{Fubini}$$

$$= \iint_D \big[R(x,y,u_2(x,y)) - R(x,y,u_1(x,y))\big]\,dA$$



$S = S_1 \cup S_2 \cup S_3$.

S_1 the graph : $z = u_1(x,y)$.
S_2 the graph $z = u_2(x,y)$.
S_3 vertical.

On S_3, $\vec{k}\cdot\vec{n} = 0 \Rightarrow \iint_{S_3} R\vec{k}\cdot\vec{n}\,dS = 0$.

$\vec{F} = (0,0,R)$

$$\iint_{S_2} R\vec{k}\cdot\vec{n}\,dS = \iint_D R(x,y,u_2(x,y))\,dA$$

$\int(\vec{F}\cdot d\vec{S}$

$-P\frac{\partial S}{\partial x} - Q\frac{\partial g}{\partial x} + R$

$$\iint_{S_1} R\vec{k}\cdot\vec{n}\,dS = -\iint_D R(x,y,u_1(x,y))\,dA$$

(downward orientation).

□

Example $\iint_S \vec F \cdot d\vec S$, $\vec F = z\vec i + y\vec j + x\vec k$

$S: x^2+y^2+z^2=1$

Solution: $\text{div}\vec F = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(x) = 1$

The unit sphere S is the boundary of the unit ball

$$B = \{(x,y,z) \mid x^2+y^2+z^2 \le 1\}.$$

$$\iint_S \vec F \cdot d\vec S = \iiint_B \text{div}\vec F \, dV = \underline{\text{Vol}(B) = \frac{4\pi}{3}}$$

□

$$\begin{tikzcd}[column sep=large]
\text{scalar field} \arrow[r, "\nabla"] \arrow[r, bend right=40, "\substack{\text{Fundamental theorem}\\\text{of Calculus for line integrals}}"'] & \text{vector field} \arrow[r, "\nabla\times \,\doteq\, \text{curl}"] \arrow[r, bend right=40, "\substack{\text{Green's theorem (dim2)}\\\text{Stokes' theorem (dim3)}}"'] & \text{vector field} \arrow[r, "\overset{\text{div}}{\nabla\cdot}"] \arrow[r, bend right=40, "\substack{\text{divergence}\\\text{theorem}}"'] & \text{scalar field}
\end{tikzcd}$$

$\int_P^Q \nabla f \cdot d\vec r = f(Q) - f(P)$

$\iint_S \text{curl}\,\vec F \cdot d\vec S = \int_{\partial S} \vec F \cdot d\vec r$

$\iiint_E \text{div}\vec F \, dV = \iint_S \vec F \cdot d\vec S$

("differential of field")

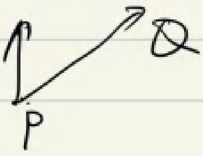
boundary of domain)

2, physical interpretations of field theory.

① line integral (of vector fields) $\int_C \vec F \cdot d\vec r$.

The work done by a constant force $\vec F$ in moving an object from a point P to another point Q in space is.

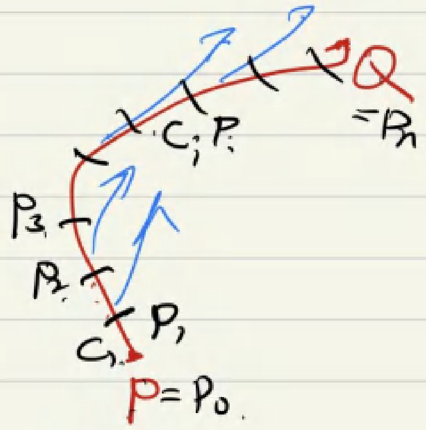
$$W = \vec F \cdot \vec D$$



where $\vec D = \overrightarrow{PQ}$ is the displacement vector.

Now let $\vec F$ be a force field,

C a curve from P to Q.



We divide C into many pieces

$P_0=P, \cdots P_i \cdots\cdots \qquad P_n=Q,$

Along each piece $\overset{=\widehat{P_{i-1}P_i}}{C_i}$ $\vec F \approx \vec F(P_i^*)$, where P_i^* is any sample point in C_i. So the work of $\vec F$ done in moving a particle from P_{i-1} to P_i is

$$\vec F(P_i^*)\cdot\overrightarrow{P_{i-1}P_i} \approx \vec F(P_i^*)\cdot \underset{\text{direction}}{\vec T(P_i^*)}\underset{\text{length } \overline{P_{i-1}P_i}}{ds}.$$

Take Riemann sum and then take limit,

Line integral $\Longleftrightarrow$ work of a force field done in moving a particle along the curve.

② surface integral (of vector fields)

(Hat).

Suppose S is a rectangle of area ΔS.

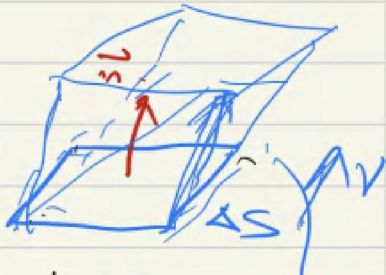


Image a fluid with constant velocity $\vec v$. flowing through S. After a unit time, the volume flowing through S is the volume of the parallelepiped:

$$V = \underline{\Delta S}\cdot\underline{\vec v}\cdot\underline{\vec n},$$

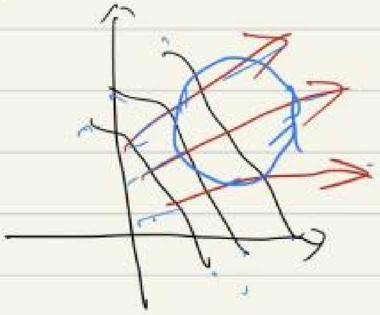
Now we replace S by a surface, $\vec v$ by a vector field $\vec F$, $\Delta S\cdot\vec v\cdot\vec n$ by the Riemann sum.

$$\sum \vec F\cdot\vec n\,\Delta S_{ij}$$

Then take limit,

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\,dS.$$

is the volume of the fluid with velocity field $\vec{F}$ flowing through S per unit time.



③ Gradient

potential function ⟶ force field.

Example : electric potential ⟶ electric field,

The electric potentials tells you the work the electric field done in moving a particle from a point P to Q.

$$\int_P^Q \underbrace{\nabla f\cdot d\vec{r}}_{\text{work}} = \underbrace{f(Q) - f(P)}.$$

potential functions

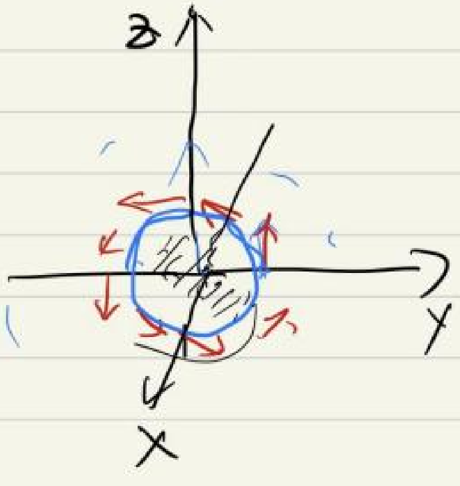
④ curl:

Curl measures the tendency of the fluid to swirl around the point. The magnitude of the curl measures how much the fluid is swirling, the direction indicates the axis around which is tends to swirl.

Example $\vec{F} = -y\vec{i} + x\vec{j}$.

$\text{curl}\,\vec{F} = 2\vec{k}$

$\vec{F}$ rotates around the z-axis



This interpretation can be seen from Stokes' theorem.

$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_{\partial S} \vec{F}\cdot d\vec{r}.$$

∇f does not swirl, so $\text{curl}(\nabla f) = \vec{0}$

⑤ div

divergence measures the tendency of the fluid to collect or disperse at point.;

$$\iiint_E \text{div}\vec{F}\,dV = \iint_{\partial E} \vec{F}\,d\vec{S}.$$

Take a small ball around a point p.

swirls

curl $\vec{F}$ does not collect or disperse, so $\text{div}(\text{curl}\,\vec{F}) = 0$.

⑥

	electric field	magnetic field.
curl	$=0.$ (potential)	$\text{curl}\,\vec{B} = \frac{4\pi\vec{J}}{c}$ $\vec{J}$: current density
div	Gauss's theorem $\int \vec{E}\cdot d\vec{S} = 4\pi Q$ $\Leftrightarrow \text{div}\,E = 4\pi\rho$ charge	$=0.$ magnetic monopoles do not exist

⑦ Fundamental theorem of vector calculus.
(Helmholtz decomposition)

A vector field (satisfying appropriate smoothness and decay conditions) can be decomposed as the sum of the form

$$\underline{\underline{-\nabla\phi}} + \underline{\nabla\times\vec{A}}.$$

3, de Rham cohomology. $d=0 \Longleftrightarrow$ image of d. "locally"

(1) $\vec{F}=\nabla f \Rightarrow \operatorname{curl}\vec{F}=0$.

$\operatorname{curl}\vec{F}=\vec{0}$ $\quad\vec{F}$ / simply connected domain (say, a ball).

$\Rightarrow \vec{F}=\nabla f$. $\quad\left(\int_P^Q \vec{F}\cdot d\vec{r} = f(Q)-f(P)\right)$.

$\vec{F}=\operatorname{curl}\vec{G} \Rightarrow \operatorname{div}\vec{F}=0$

$\operatorname{div}\vec{F}=0$ $\quad\vec{F}$ / simply connected domain

$\Rightarrow \vec{F}=\operatorname{curl}\vec{G}$. $\quad$ (Poincaré lemma)

(2) If D is NOT simply connected.

Let $D=\mathbb{R}^2\setminus 0$

$$\vec{G}=\frac{-y}{x^2+y^2}\vec{i}+\frac{x}{x^2+y^2}\vec{j}, = \text{"}\underline{\nabla \arctan\frac{y}{x}}\text{"}$$

"$\theta=\arctan\frac{y}{x}$ is NOT a real function"

Let C be a positively oriented simple closed path

$$\int_C \vec{G}\,d\vec{r} = \begin{cases} 2\pi & \text{If C } \underline{\text{encloses the origin}}. \\ 0 & \text{if NOT.} \end{cases}$$

Now let $\vec{F}$ be any vector field over D, such that

$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0$.

If $\int_C \vec{F}\cdot d\vec{r} = k$ (C encloses the origin),

$\int_C\left(\vec{F}-\frac{k}{2\pi}\vec{G}\right)d\vec{r}=0$ for <u>any positively oriented simple closed path</u>

So $\vec{F} - \frac{k}{2\pi}\vec{G} = \nabla f$ for some function f.

Example $\vec{F} = \left\langle \frac{y^3+x^2y-4x}{(x^2+y^2)^2}, -\frac{(xy^2+4y+x^3)}{(x^2+y^2)^2} \right\rangle$

Solution: Check that curl $\vec{F} = 0$. $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0\right)$.

Let C be the unit circle:

$\vec{r}(t) = (\cos t, \sin t) \quad 0 \le t \le 2\pi,$

$\vec{r}'(t) = (-\sin t \ \cos t).$

$$\vec{F}(\vec{r}(t)) = (\sin^3 t + \cos^2 t \sin t - 4\cos t)(-\sin t) - (\cos t \sin^2 t + 4\sin t + \cos^3 t)(\cos t)$$

$$= -(\cos^4 t + \sin^4 t + 2\cos^2 t \sin^2 t)$$

$$= -(\cos^2 t + \sin^2 t)^2 = -1$$

$\int_C \vec{F}\cdot d\vec{r} = -2\pi.$

$\vec{F} + \vec{G} = \left\langle \frac{-4x}{(x^2+y^2)^2}, \frac{-4y}{(x^2+y^2)^2} \right\rangle,$

$\vec{F} + \vec{G} = \underline{\nabla\left(\frac{2}{x^2+y^2}\right)}.$

□

$f /$ over $\mathbb{R}^2 \setminus 0$

Let K be the vector space of all vector fields $\vec{F}$ such that $\text{curl}\,\vec{F} = \vec{0}$,

E be the vector space of all conservative vector fields.

K/E be their quotient space.

Then $\int_C \vec{F}\cdot d\vec{r}$. defines a linear map.

$$K/E \longrightarrow \mathbb{R},$$

<say, the unit circle>

where C is any positively oriented simple closed curve, the encloses the origin

- $\int_C \vec{F}\cdot d\vec{r} = 0$ if $\vec{F} \in E$, so well-defined.
- $\int_C$ is surjective $c\vec{G} \to 2\pi c$.
- $\int_C$ is injective :

  if $\int_C \vec{F} - \vec{H} = 0$. $\vec{F} - \vec{H} = \nabla f$ for some f.

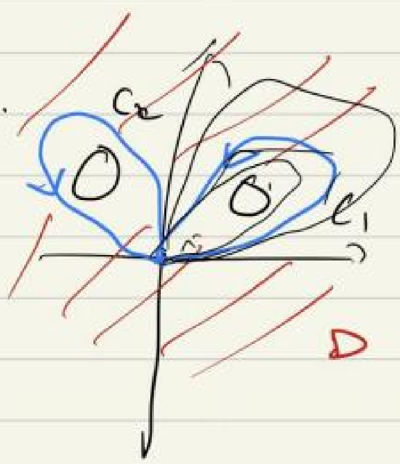
In summary, the gap between a vector field with $\text{curl}\,\vec{F} = \vec{0}$ and a conservative field is measured by its integration along the unit circle. (or any p.o s.c. encloses the origin)

Actually, we regrad all simple closed curves enclosing the origin as "the same.". We may use $[C]$ to represent such a class of curves.

$$(\text{curl}\,\vec{F} = 0) \dashrightarrow \int_C \vec{F}\cdot d\vec{r} = \int_{C'} \vec{F}\cdot dr$$

$\int \vec{F} \cdot d\vec{r}$

For more complicated, region,
you need to evaluate the integrals
of $\vec{F}$ (curl $\vec{F} = 0$) along more
closed curves,



These curves (actually the classes of curves),

measures the complexity of the region, They "represent" holes in **D**.

A vector field $\vec{F}$ (curl $\vec{F}$) is conservative

$$\Longleftrightarrow$$

$$\int_{C_i} \vec{F} \cdot d\vec{r} = 0.$$

HE

(functions, differential operators) analytically

The quotient of vector fields (curl $\vec{F} = \vec{0}$) modulo conservative field $\Longleftrightarrow$ de Rham cohomology

$\Updownarrow$ $\Updownarrow$

linear functions of classes of certain closed curves $\Longleftrightarrow$ singular cohomology

HE: $(C_1 \ C_2) \longrightarrow \left(\int_{C_1} \vec{F} \cdot \int_{C_1} \vec{F} \right)$

(chains boundaries)

topologically

What about vector fields $\vec{F}$ (div $\vec{F} = 0$).

Integrate over closed surfaces.

The classes of surfaces represent "holes"

Example. $\mathbb{R}^3 - 0$: S^2.

$\square$

$$(\operatorname{div}\vec{F} = 0) \Big/ \operatorname{curl}\vec{G} \simeq H^2(D)$$

Lecture 24. Review

1. Preparations.

· Geometry.

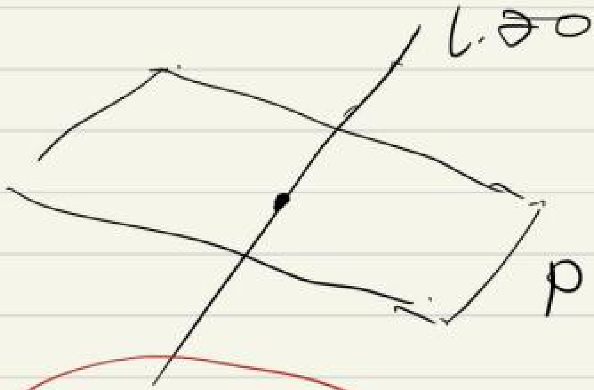
$\mathbb{R}^3$. dot produce $\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cdot\cos\theta$.

cross produce $\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$.

↓

orthogonal to $\vec{a}$, $\vec{b}$.

"dual" lines ⟷ planes.



general lines ⇕ translation.

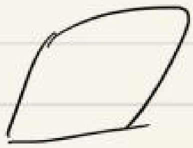
(a,b,c) → direction vector of a line
→ normal vector of a plane.

$\vec{r} = \vec{r_0} + t\vec{v}$

$Ax + By + Cz = d$.

interval. $\in \mathbb{R} \longrightarrow \vec{r}(t)$

curve.



$D \in \mathbb{R}^2 \longrightarrow \vec{r}(u,v)$

parametric.

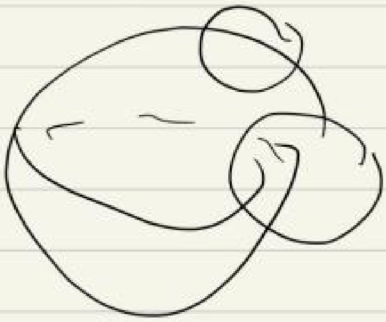
↓

graphs

level surfaces.

locally it is a parametric surface

(implicit function theorem)



. Calculus study general things

functions /. curve ⟶ linear maps

lines/planes

direction

tangent line $\vec{r}(t) \longrightarrow r'(t)$

tangent plane $\vec{r}(u,v) \longrightarrow \vec{r_u}\times\vec{r_v}$

- Algebra.

Matrices: $\left(\qquad \right)_{m\times n}$

$M_{m\times n}(\mathbb{R}) \longleftrightarrow$ linear map $\mathbb{R}^n \longrightarrow \mathbb{R}^m$.

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \to \left(\qquad \right)_{m\times n} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

matrix multiplication $\longleftrightarrow$ composition of linear maps.

determinant: "volume element", $\det(J(f))$

$\longrightarrow \left|\ \ \right| =$ "sign area" of

depends on the orientation

$$\longrightarrow \begin{vmatrix} a_1 & a_2 & a_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \text{"}\underset{\uparrow}{\text{signed}}\text{ volume"}$$

2. differentials:

- $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

  $p \longrightarrow f(p)$.

  $p \to p+\Delta p \longrightarrow f(p+\Delta p) - f(p)$

  $= \underline{(\qquad)}\Delta p + h.o.t.$

  $= J(F)_p$.

  $J(F)_p \longrightarrow$ linear approximation of f.

  ↑ ↑

  properties - - → local properties of f.

  { inverse function theorem
  { implicit function theorem --

- J ⟶ linearization.

J of composition of functions

⇕

the chain rule ← composition ~~of~~ J's.

⇕

matrix multiplication

To study local minimum / maximums

- critical point $\nabla f|_p = \vec{0}$.
- second derivatives test.

$$\boxed{y = \underline{\sum a_{ij} x^i y^j}} \longrightarrow$$

if $\overset{\det}{H(f)} \neq 0$.

$\swarrow \begin{cases} \text{ellipetic point} & > 0 \\ \text{hyperbolic p.} & < 0. \end{cases}$

local max/min

hyperbolic p. ↓ NOT.

Principle.

linear deg1 ↓

" Some simple linear algebra (lines

quadratic forms) reflects the local properties

↑ deg2

of f.

" derivaties w.r.t one varible "

\+ " linear algebra"

↓ — several variables.

$\longrightarrow \int_a^b f\,dx$

3, Integration

$\iint f(x,y)\,dA$

$\iiint f(x,y,z)\,dV.$ $\Big\}\xRightarrow{\text{Fubini}}$ iterated integrals.

$\int_C f\,ds \qquad \int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\vec{T}\,ds.$

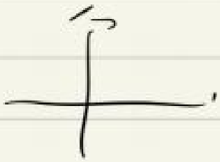
$\iint_S f\,dS \qquad \iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\,dS.$

$\int_a^b \qquad \iint_D$ integrals / region in $\mathbb{R}/\mathbb{R}^2$

$\boxed{\text{change of variables.}}$

$\searrow$ simplifies the D.

disk



$\int_0^1\int_{\sqrt{1-x^2}}^{\sqrt{1-x^2}} \cdots dy\,dx$

$\Downarrow$ polar coordinate

$dA = |\quad|\,du\,dv$

a. $|\det J|$

$\boxed{\int_0^{2\pi}\int_0^1}$ $g(r)h(\theta)$

$(\int h)(\int g)$

4 Fundamental theorems.

$$\begin{tikzcd} \text{scalar field} \arrow[r, "\nabla", "\text{grad}"'] & \text{vector field.} \arrow[r, "\nabla\times", "\text{curl}"'] & \text{vector field} \arrow[r, "\nabla\cdot", "\text{div}"'] & \text{scalar field} \end{tikzcd}$$

∇: $\int_P^Q \nabla f \cdot d\vec{r} = f(Q) - f(P).$

$\nabla\times$: $\iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}.$

(Green's. Stokes'.)

$\nabla\cdot$ $\iiint_E \text{div}\,\vec{F}\, dV = \iint_{\partial E} \vec{F} \cdot d\vec{S}$

analytically → integral of differential of some field. over domain.

$\Updownarrow$.

"dual" in some sense.

~~topologicall~~ → integral of some field over the boundary of the domain
geometrically

□

8.

scalar/
vector fields ⟶ differential forms

∇, $\nabla\times$, $\nabla\cdot$ ⟶ exterior differential: d.

$$\boxed{\int_D d\omega = \int_{\partial D} \omega}$$

- advanced calculus?
- Differential manifold.?
- Differential geometry.?

□