THE ARTHUR-SELBERG TRACE FORMULA AND CLASSIFICATION OF AUTOMORPHIC REPRESENTATIONS

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> Go down deep enough into anything
> and you will find mathematics.
>
> Dean Schlicter

The trace formula is one of the most powerful methods so far in studying the Langlands program. It describes the character of the representation of $G(\mathbb{A}_F)$ on the discrete part $L^2_0(G(F)\backslash G(\mathbb{A}_F))$ of $L^2(G(F)\backslash G(\mathbb{A}_F))$ in terms of geometric data, where G is a reductive algebraic group defined over a global field F and $\mathbb{A}_F$ is the ring of adeles of F. We usually calculate the geometric side to get information on the mysterious spectral side. Applications of the trace formula include the functoriality principle and classification of automorphic representations. The first three sections are essentially a summary of Arthur's lecture notes [Art4].

1. The Trace Formula

Let G be an algebraic group defined over $\mathbb{Q}$. We are mainly interested in $L^2(\Gamma\backslash X)$ where Γ is a discrete subgroup of $G(\mathbb{R})$, or $\Gamma\backslash X = G(F)\backslash G(\mathbb{A})$. As a general principle in representation theory, the study of representations is essentially the study of characters, or, the traces of operators. The Selberg-Arthur trace formula describes the character of the representation of $G(\mathbb{A})$ on the discrete part $L^2_0(G(F)\backslash G(\mathbb{A}))$ of $L^2(G(F)\backslash G(\mathbb{A}))$ in terms of geometric data. In most cases of Selberg-Arthur trace formula, the quotient $G(F)\backslash G(\mathbb{A})$ is not compact, which causes the following problems:

(a). *Spectral side:* The representation on $L^2(G(F)\backslash G(\mathbb{A}))$ contains not only discrete components but also continuous components. So we need a good description of the spectrum decomposition.

(b). *Geometric side:* The kernel is no longer integrable over the diagonal, and the operators $R(f)$ are no longer of trace class. We have to modify divergent integrals to make them converge. Note that traces should be interpreted as distributions, not simply functions.

Another method is to just considering simple functions to avoid the difficulties mentioned above. This is the so-called *simple trace formula*. The formula is easier to calculate in practice, but it will be less powerful theoretically. We will not discuss this here.

[Lap] is a short introduction to the trace formula. [Whi1] is an introduction to the trace formula for GL_2. Some functional analysis nonsense. [Art4] is an excellent introduction to trace formulas and their applications in automorphic representations.

Date: October 3, 2023.

1.1. **Compact case.** In this subsection, we assume that G is defined over $\mathbb{Q}$. $G(\mathbb{R})$ is then a Lie group. We use G to stand for this Lie group to simplify notations. Fix a Haar measure dg of G. Let Γ be a cocompact arithmetic subgroup of G. The Haar measure descends to a Haar measure dx on $\Gamma\backslash G$ by the left-invariance of dg. We consider the regular representation R of G on $L^2(\Gamma\backslash G)$:

$$[R(g)]\phi(x) = \phi(xg), g \in G, x \in \Gamma\backslash G.$$

This extends to a representation of the convolution algebra $C_c^\infty(G)$ [1] :

$$[R(f)]\phi(x) = \int_G f(g)\phi(xg)dg = \int_G f(x^{-1}g)\phi(g)dg^2$$

The operator $R(f)$ is of trace class, and we want to compute $\mathrm{tr}(R(f))$. There are two methods to compute it:

(1) Geometric method. If $f \in C_c^\infty(G)$, the operator $R(f)$ has a kernel $K_f(x,y) = \sum\limits_{\gamma\in\Gamma} f(x^{-1}\gamma y)$:

$$[R(f)]\phi(x) = \int_G f(x^{-1}g)\phi(g)dg = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)\phi(y)dy.$$

$\mathrm{tr}(R(f))$ is just the integral of the kernel over the diagonal (the trace of an "infinite dimensional" matrix is the "sum" over the diagonal):

$$\mathrm{tr}(R(f)) = \int_{\Gamma\backslash G} K_f(x,x)dx = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(x^{-1}\gamma x)dx.$$

We can break the sum over γ into conjugacy classes of Γ. The conjugacy class

$$[\gamma] = \{\delta^{-1}\gamma\delta : \delta \in \Gamma_\gamma\backslash\Gamma\}$$

where Γ_γ is the centralizer of γ in Γ, contributes

$$\int_{\Gamma\backslash G} \sum_{\delta\in\Gamma_\gamma\backslash\Gamma} f(x^{-1}\delta^{-1}\gamma\delta x)dx = \int_{\Gamma_\gamma\backslash G} f(x^{-1}\gamma x)dx = \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)I(\gamma, f),$$

where $I(\gamma, f)$ is the *orbital integral*

$$I(\gamma, f) = \int_{G_\gamma\backslash G} f(x^{-1}\gamma x)dx.$$

In summary,

$$\mathrm{tr}(R(f)) = \sum_{[\gamma]} \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)I(\gamma, f)^3. \tag{1}$$

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[1]Formally, $R(f) = \int_G R[g]f(g)dg$. It is the weighted integration of the operator $R(g)$ with respect to the measure $f(g)dg$. We can also define $R(d\mu) = \int_G R[g]d\mu(g)$ for a bounded measure $d\mu$.

[2]Of course, the $\phi(g)$ in the third term means the lifting to G under the natural projection $p : G \to \Gamma\backslash G$.

[3]Clearly, the volume and the orbital integral are all defined on conjugacy classes.

(2) Spectral method. First recall that we have a very simple description of the regular representation in the compact case:

Proposition 1.1 (Gelfand-Graev-Piatetski-Shapiro)**.** *$L^2(\Gamma\backslash G)$ decomposes discretely into a direct sum of irreducible representations of G, each occuring with finite multiplicity* [4].

Let $\hat{G}$ be the unitary dual of G, from the representation decomposition

$$R = \sum_{\pi\in\hat{G}} m(\pi)\pi,$$

we get another computation of the trace

$$\mathrm{tr}(R(f)) = \sum_{\pi\in\hat{G}} m(\pi)\mathrm{tr}\ \pi(f). \tag{2}$$

Remark 1.1. The trace map $\mathrm{tr}(\pi) : C_c^\infty(G(F)) \to \mathbb{C}$ defines a distribution. Let $G^{reg} \leq G$ denote the subscheme consisting of regular semisimple elements; this is the subscheme such that

$$G^{reg}(R) = \{\gamma \in G(R) : C_\gamma^\circ \text{ is a maximal torus}\}.$$

The Harish-Chandra theorem asserts that the distribution $\mathrm{tr}(\pi)$ is represented by a locally constant function with support in $G^{reg}(F)$.

Then the *Selberg trace formula* says that the geometric trace (1) and the spectral trace (2) are equal:

$$\sum_{[\gamma]} \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)I(\gamma, f) = \sum_{\pi\in\hat{G}} m(\pi)\mathrm{tr}\ \pi(f).$$

In particular, if we set $G = \mathbb{R} = \mathbb{G}_a(\mathbb{R})$ and $\Gamma = \mathbb{Z}$, we recover the Poisson summation formula:

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n), f \in C_c^\infty(\mathbb{R}). \tag{3}$$

1.2. **Arthur's trace formula.** Now we want to study the right regular representation on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ for a general algebraic group over $\mathbb{Q}$. The space $G(\mathbb{Q})\backslash G(BA)$ is not compact, which makes the trace formula difficult. Let f be a smooth function of compact support, now the kernel operator is not of trace class. As pointed out in [Art4], the divergence comes from parabolic subgroups. Fix a minimal parabolic subgroup P_0. We truncate the kernel function K_f by an alternating sum of functions parameterized by standard parabolic subgroups. The truncation depends on a parameter $T \in i\mathfrak{a}_P^*$ and is defined for sufficiently regular T. After the truncation, K_f^T is "supported in a large compact subset". So we may compute the geometric and spectral side of this kernel function. We, therefore, get a family of equations, depending on T.

But this is not what we really want. We need something intrinsic, that is, independent of our truncation parameter T. It turns out that both sides of the equations are

[4] If Γ is the trivial group, this is (part of) the Peter-Weyl theorem.

polynomials of T. Therefore, the "constant term" could be an intrinsic formula. This is the trace formula we get:

Theorem 1.2 (Arthur's trace formula)**.**

$$\sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}(f) = \sum_{\chi\in\mathcal{X}} J_{\chi}(f), \qquad f \in C_c^{\infty}(G(\mathbb{A})) \tag{4}$$

We will explain the terms and their calculations in the following subsections. We may calculate the terms in the trace formula explicitly by calculating explicitly the truncated trace formula and then take constant coefficients. But this only works for very simple groups. On the other hand, for general groups, the generic term in the formula is a weighted orbital integral or a weighted character.

1.2.1. *Truncations.* The kernel $k(x,x)$ does not converge, so we need to truncate it to male it converge. The truncation depends on a fixed minimal parabolic subgroup P_0 and a regular element $T \in \mathfrak{a}_0^+$. The truncation is a sum over standard parabolic subgroups relative to P_0. So we first recall some standard definitions.

Let G be an algebraic group over $\mathbb{Q}$. We write A_G for the largest central subgroup of G over $\mathbb{Q}$ that is a $\mathbb{Q}$-split torus. The rank of A_G is called the rank of G. We write $X(G)_{\mathbb{Q}}$ for the additive group of homomorphisms $\chi : g \to g^{\chi}$ from G to $GL(1)$ that are defined over $\mathbb{Q}$. Then $X(G)_{\mathbb{Q}}$ is a free abelian group of rank k. WE also form the real vector space

$$\mathfrak{a}_G = \mathrm{Hom}_{\mathbb{Z}}(X(G)_{\mathbb{Q}}, \mathbb{R})$$

of dimension k. There is then a surjectuve homomorphism $H_G : G(\mathbb{A}) \to \mathfrak{a}_G$, defined by

$$\langle H_G(x), \chi\rangle = |\log(x^{\chi})|, \quad x \in G(\mathbb{A}), \chi \in X(G)_{\mathbb{Q}}.$$

A *parabolic subgroup* of G is a $\mathbb{Q}$-algebraic subgroup P such that $P(\mathbb{C})\backslash G(\mathbb{C})$ is compact. Any such P has a Levi decomposition $P = MN_P$, which is a semidirect product of a reductive subgroup M of G over $\mathbb{Q}$ with a normal unipotent subgroup N_P of G over $\mathbb{Q}$. The *unipotent radical* N_P is uniquely determined by P, while the *Levi component* M is uniquely determined up to conjugacy by $P(\mathbb{Q})$.

Let P_0 be a fixed parabolic subgroup of G with a fixed Levi decomposition $P_0 = M_0N_0$. Any subgroup P that contains P_0 is called a *standard parabolic subgroup*. The set of standard parabolic subgroup is finite, and is a set of representatives of the set of $G(\mathbb{Q})$-conjugacy classes of parabolic subgroups over $\mathbb{Q}$. A standard parabolic subgroup has a canonical Levi decomposition $P = M_PN_P$, where M_P is the unique Levi component of P that contains P_0. From M_P, we can form the central subgroup $A_P = A_{M_P}$, the real vector space $\mathfrak{a}_P = a_{M_P}$, and the surjective homomorphism $H_P = H_{M_P}$. When $P = G$, we recover the original definition of A_G, $\mathfrak{a}_G$, and H_G. If $P = P_0$, we use the notations A_0, $\mathfrak{a}_0$, and H_0. We extend H_P to a fucntion from $G(\mathbb{A})$ by setting $H_P(nmk) = H_{M_P}(m)$.

We have a variant of the regular representation R for any standard parabolic subgroup P. It is the regular representation R_P of $G(\mathbb{A})$ on $L^2(N_P(\mathbb{A})M_P(\mathbb{Q})\backslash G(\mathbb{A}))$. It is the induced representation

$$R_P = \mathrm{Ind}_{N_P(\mathbb{A})M_P(\mathbb{Q})}^{G(\mathbb{A})}(1_{N_P(\mathbb{A})M_P(\mathbb{Q})}) \cong \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(1_{N_P(\mathbb{A})} \otimes R_{M_P}).$$

$R_P(f)$ is an integral operator with kernel

$$K_P(x,y) = \int_{N_P(\mathbb{A})} \sum_{\gamma \in M_P(\mathbb{Q})} f(x^{-1}\gamma n y) dn, x, y \in N_P(\mathbb{A}) M_P(\mathbb{Q}) \backslash G(\mathbb{A}).$$

For a given standard parabolic subgroup P, we may define two subsets of $\mathfrak{a}_P$: the set of roots Δ_P and the set of weights $\hat{\Delta}_P$. We write τ_P for the characteristic function of the subset

$$\mathfrak{a}_P^+ = \{t \in \mathfrak{a}_P : \alpha(t) > 0, \alpha \in \Delta(P)\}$$

of $\mathfrak{a}_P$. We also write $\hat{\tau}_P$ for the characteristic function of the subset

$$\{t \in \mathfrak{a}_P : \varpi(t) > 0, \varpi \in \hat{\Delta}_P\}.$$

The truncation of $K(x,x)$ depends on a parameter T in the cone $\mathfrak{a}_0^+$ that is suitably regular. For any given T, we define

$$k^T(x) = k^T(x,f) = \sum_P (-1)^{\dim A_P/A_G} \sum_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})} K_P(\delta x.\delta x)\hat{\tau}_P(H_P(\delta x) - T).$$

It is a well-defined function of $x \in G(\mathbb{Q})\backslash G(\mathbb{A})$.

Theorem 1.3. *The integral*

$$J_f^T = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} k^T(x,f) dx$$

converges absolutely.

The convergent integrals are not what we really want as they depend on a parameter T. To get intrinsic trace formula, we need to study the behavior of the truncated integrals with respect to the parameter T.

Theorem 1.4. *For any $f \in C_c^\infty(G(\mathbb{A}))$, the function*

$$T \mapsto J_f^T,$$

defined for $T \in \mathfrak{a}_0^+$ sufficiently regular, is a polynomial in T whose degree is bounded by the dimension of $\mathfrak{a}_0^G$.

From the above theorem, we may expect that the constant coefficients, or higher coefficients, could be a candidate for the intrinsic trace formula. It turns out that there is a better choice in general. The question is related to the choice of minimal parabolic subgroup P_0

In the definition of truncation functor, we fixed a minimal parabolic subgroup P_0. Fix M_0, and let write $\mathcal{P}(M_0)$ for the set of minimal parabolic subgroups of G with Levi component M_0. If we change the choice of P_0 in $\mathcal{P}_{M_0}$, we get different polynomials in $\mathfrak{a}_0$. However, there is a unique point T_0 (depending on K), such that the value of polynomials is independent of the choice of P_0. The value at this point is the real "constant term". If $G = GL(n)$ with the canonical choice of K, the point T_0 is zero, as expected.

We define $J(f)$ as the value of $J^T(f)$ at the point T_0, $f \mapsto J(f)$ is a distribution on $G(\mathbb{A})$.

Remark 1.2. The distribution J satisfies the formula

$$J(f^y) = \sum_{Q \supset P_0} J^{M_Q}(f_{Q,y})$$

for conjugation of $f \in C_c^\infty(G(\mathbb{A}))$ by $y \in G(\mathbb{A})$: $f^y(x) = f(yxy^{-1})$. The transformation $f \to f_{Q,y}$ is a continuous linear mapping from $C_c^\infty(G(\mathbb{A}))$ to $C_c^\infty(M_Q(\mathbb{A}))$ defined in [Art4], P53. In general, the trace formula is not invariant over conjugacy, so we need a stable trace formula.

The next step is to study the distribution J in geometric and spectral methods, and the equality of the two expansions is the trace formula.

1.2.2. *The coarse geometric expansion.* We want to get a geometric expansion of $J(f)$ from the geometric expansion of $J^T(f)$. However, the truncation $k^T(f)$ of $K(x,x)$ is not completely compatible with the decomposition of $K(x,x)$ according to conjugacy classes. We need a weaker conjugacy class.

We define two elements γ and γ' in $G(\mathbb{Q})$ to be $\mathcal{O}$-equivalent if their semisimple parts γ_s and γ'_s are $G(\mathbb{Q})$-conjugate. We then write $\mathcal{O}$ for the set of such equivalence classes in $G(\mathbb{Q})$. A class $\mathfrak{o} \in \mathcal{O}$ is thus a union of conjugacy classes in $G(\mathbb{Q})$. The set $\mathcal{O}$ is in obvious bijection with the semisimple conjugacy classes in $G(\mathbb{Q})$.

We say that a semisimple conjugacy class in $G(\mathbb{Q})$ is *anisotropic* if it does not intersect $P(\mathbb{Q})$ for any $P \subset G$. Then $\gamma \in G(\mathbb{Q})$ represents an isotropic class if and only if A_G is the maximal $\mathbb{Q}$-split torus in the connected centralizer H of γ in G. We define an *anisotropic rational datum* to be an equivalence class of pairs (P, α), where $P \subset G$ is a standard parabolic subgroup, and α is an anisotropic conjugacy class in $M_P(\mathbb{Q})$. The equivalence relation is just conjugacy. The mapping sends $\{(P,\alpha)\}$ to the conjugacy class α in $G(\mathbb{Q})$ is a bijection onto the set of semisimple conjugacy classes in $G(\mathbb{Q})$.

Example 1.5. *In case $G = GL(n)$, the $\mathcal{O}$-equivalence classes are parameterized by the set of complex eigenvalues or characteristic polynomials.*

Under this coarser equivalence class, $k^T(x) = \sum_{\mathfrak{o} \in \mathcal{O}} k^T_{\mathfrak{o}}(x)$, where

$$k^T_{\mathfrak{o}}(x) = k^T_{\mathfrak{o}}(x,f) = \sum_P (-1)^{\dim A_P/A_G} \int_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})} K_{P,\mathfrak{o}}(\delta x, \delta x) \hat{\tau}_P(H_P(\delta x) - T).$$

This decomposition induces decomposition

$$J^T(f) = \sum_{\mathfrak{o} \in \mathcal{O}} J^T_{\mathfrak{o}}(f).$$

The convergence and growth conditions of $J^T_{\mathfrak{o}}(f)$ can be obtained similarly to that of $J^T(f)$. Evaluating at the point T_0, we get the *coarse geometric expansion*:

$$J(f) = \sum_{\mathfrak{o} \in \mathcal{O}} J_{\mathfrak{o}}(f), \qquad f \in C_c^\infty G(\mathbb{A}).$$

Remark 1.3. For each $\mathfrak{o} \in \mathcal{O}$, we still have the relation

$$J_{\mathfrak{o}}(f^y) = \sum_{\mathfrak{o}_Q \supset P_0} J^{M_Q}_{\mathfrak{o}}(f_{Q,y})$$

where $\mathfrak{o}_Q$ ranges over the finite preimage of $\mathfrak{o}$ in $\mathcal{O}^{M_Q}$ under the obvious mapping of $\mathcal{O}^{M_Q}$ into $\mathcal{O} = \mathcal{O}^G$.

If $\mathfrak{o}$ is anisotropic, it does not lie in the image of the map $\mathcal{O}^{M_Q} \to \mathcal{O}$ attached to any proper parabolic subgroup $Q \subsetneq G$. The distribution $J_{\mathfrak{o}}$ is invariant in this case.

What are the integrals $J_{\mathfrak{o}}(f)$? We would expect, as in the compact case, they are products of volumes and orbital integrals. This is too optimistic, but is true for a generic class, just replacing the orbital integrals with *weighted orbital integrals*. Suppose that (P, α) represents the anisotropic rational datum attached to $\mathfrak{o}$, and γ belongs to the anisotropic conjugacy class α in $M_P(\mathbb{Q})$. We say that $\mathfrak{o}$ is *unramified* if $G(\mathbb{Q})_\gamma$ is also contained in M_P. This is equivalent to asking that the centralizer of the conjugacy class α in $W(\mathfrak{a}_P, \mathfrak{a}_P)$ be equal to $\{1\}$. In the case $G = GL(n)$, this condition is automatically satisfied.

Theorem 1.6. *Suppose that $\mathfrak{o} \in \mathcal{O}$ is an unramified class, with anisotropic rational datum represented by a pair (P, α). Then*

$$J_{\mathfrak{o}}(f) = \mathrm{vol}(M_P(\mathbb{Q})_\gamma \backslash M_P(\mathbb{A})^1_\gamma) \int_{G(\mathbb{A})_\gamma \backslash G(\mathbb{A})^1} f(x^{-1}\gamma x) v_P(x) dx,$$

where γ is any element in the $M_P(\mathbb{Q})$-conjugacy class α, and $v_P(x)$ is the volume of some set depending on x, see [Art4].

1.2.3. *The coarse spectral expansion.* We define a *cuspidal automorphic data* to be an equivalence class of pairs (P, σ), where $P \subset G$ is a standard parabolic subgroup of G, and σ is an irreducible representation of $M_P(\mathbb{A})^1$ occurring in $L^2_{cusp}(M_P(\mathbb{Q}) \backslash M_P(\mathbb{A})^1)$. The equivalence relation is defined by conjugacy. We write $\mathcal{X}$ for the set of cuspidal automorphic date $\chi = \{(P, \sigma)\}$.

For any P, we find a orthogonal basis $\mathcal{B}_P$ of $\mathcal{H}_P$ with is compatible with χ-decompositon: $\mathcal{B}_P = \coprod \mathcal{B}_{P,\chi}$. For any $\chi \in \mathcal{X}$, let

$$K_\chi(x, y) = \sum n_P^{-1} \int_{i\mathfrak{a}_P^*} \sum_{\phi \in \mathcal{B}_{P,\chi}} E(x, \mathcal{I}_{P,\chi}(\lambda, f)\phi, \lambda) \overline{E(y, \phi, \lambda)} d\lambda.$$

$K_\chi(x, y)$ is the kernel of the restriction of $R(f)$ to the invariant subspace $L^2_\chi(G(\mathbb{Q}) \backslash G(\mathbb{A}))$. So we have the decomposition

$$K(x, y) = \sum K_\chi(x, y).$$

We then use a similar method in defining coarse geometric expansion to get $k^T(x) = \sum k^T_\chi(x)$, and a coarse spectral expansion

$$J(f) = \sum_{\chi \in \mathcal{X}} J_\chi(f), \quad f \in C_c^\infty(G(\mathbb{A}))$$

Remark 1.4. The convergence on the spectral side is more subtle. We need to define another truncation function Λ^T_2 and compare the two functors. See [Art4].

Remark 1.5. The distributions $J_\chi(f)$ are again generally not invariant. We actually have the variance property

$$J_\chi(f^y) = \sum_{Q \supset P_0} J^{M_Q}_\chi(f_{Q,y}).$$

As before, $J_\chi^{M_Q}$ is defined as a finite sum of distributions $J_{\chi_Q}^{M_Q}$, in which χ_Q ranges over the preimage of χ in $\mathcal{X}^{M_Q}$ under the mapping of $\mathcal{X}^{M_Q}$ to $\mathcal{X}$.

If χ is cuspidal, it does not lie in the image of the map $\mathcal{X}^{M_Q}$ to $\mathcal{X}$ attached to any proper parabolic subgroup $Q \subsetneq G$. The distribution J_χ is then invariant.

Again we want to find $J_\chi(f)$ explicitly, and we expect that it is a product of a multiplicity and a trace. This is almost true for a generic χ, with the modification that we need a *twisted character*.

We say a class χ is *unramified* if for every pair $(P, \sigma) \in \chi$, the stabilizer of π in $W(\mathfrak{a}_P, \mathfrak{a}_P)$ is $\{1\}$.

Theorem 1.7. *Suppose that* $\chi = \{(P, \pi)\}$ *is unramified. Then*

$$J_\chi(f) = m_{cusp}(\pi) \int_{i\mathfrak{a}_P^*} \mathrm{tr}(\mathcal{M}_P(\pi(\lambda)))\mathcal{I}_P(\pi_\lambda, f) d\lambda.$$

We have two expansions of $J(f)$, the coarse geometric expansion (LHS of 4) and the coarse spectral expansion (RHS of 4), then the trace formula is that the two expansions are the same (4).

1.3. **An example:** GL_2**.** see [Whi1] or [Cha]. The geometry and spectrum of GL_2 are simple and explicit, so we may calculate everything to get the trace formula.

The only rational character is given by det. Therefore, G^1

1.3.1. *Arthur's truncation.*

1.3.2. *Spectral decomposition.* Let $\mathcal{B}$ be an orthonormal basis of $\mathcal{H}$. The Langlands decomposition gives the following expression

$$K_{cont}(x, y) = \frac{1}{2} \int_{i\mathbb{R}} \sum_{\phi \in \mathcal{B}} E(x, R_s(f)\phi, s)\overline{E(y, \phi, s)} ds^5$$

for any $x, y \in G^1$.

1.4. **Applications.** As pointed out in [Whi1], applications of the trace formula usually fall into one of the two categories.

- Using the trace formula in isolation. One can attempt to compute the geometric expansion of the trace formula for suitable test functions f. This leads to dimension formulas for spaces of automorphic forms, closed forms for traces of Hecke operators, the existence of cusp forms, and Weyl's law, etc.
- Comparing the trace formula as one varies the group G. One can perhaps imagine trying to match up the geometric sides of the trace formula for different groups. This leads to Langlands functorialities, and decomposition of the L-function of a Shimura variety into products of automorphic L-functions when the trace formula is compared with the Lefschetz fixed point formula.

[5] This is a continuous version of the formula $\mathrm{tr}A = \sum \langle Ae_i, e_i \rangle$.

1.4.1. *Weyl's law.* Let X be a compact Riemannian manifold. The Laplacian has a discrete spectrum with nonnegative eigenvalues. Weyl's law studies the distribution of these eigenvalues. Specifically, it studies the asymptotic growth of the number of eigenvalues λ_i such that $|\lambda_i| \leq T$ when $T \to \infty$. If we study the torus $\mathbb{R}^2/\mathbb{Z}^2$, this is equivalent to counting the integral points in the discs $\{(x,y)|x^2+y^2 \leq R^2\}$ when $R \to \infty$. Now we may use the trace formula to prove Weyl's law for compact quotients $\Gamma\backslash\mathbb{H}$. See [LV] and [Whi1] for an exposition.

Theorem 1.8 (Weyl's law)**.** *Let Γ be a discrete cocompact hyperbolic subgroup of $SL_2(\mathbb{R})$. Let $0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$ denote the eigenvalues of Δ (appearing with multiplicity) acting on $\mathcal{A}(\Gamma\backslash\mathbb{H})$. For $T > 0$ we set $N_\Gamma(T) = \#\{j : \lambda_j < T\}$. Then*

$$N_\Gamma(T) \sim \frac{area(\Gamma\backslash\mathbb{H})}{4\pi}T.$$

Proof. The unitary representations of $SL_2(\mathbb{R})$ are well-known. They are discrete series, limits of discrete series, some principal series representations, complementary series, and the trivial representation. The spherical representations (non-trivial representations V with nonvanishing V^K where $K = SO(2)$ is the maximal compact subgroup) are unitary principal series representations and complementary series.

Consider the right regular representation of $SL_2(\mathbb{R})$ on $L^2(\Gamma\backslash G)$. Only subrepresentations with nonzero K-fixed vectors induce functions on $\Gamma\backslash\mathbb{H}$. The Casimir element $D \in U(\mathfrak{g}_\mathbb{C})$ acts on these representations as scalars and turns out to be the Laplacian on $\Gamma\backslash\mathbb{H}$. Now the spherical representations can be recovered from the Hecke algebra $C_c^\infty(G//K)$. By Satake isomorphism, this is isomorphic to $C_c^\infty(A)^W$ where $A \cong \mathbb{R}$ is the split torus, $W \cong S_2$ is the Weyl group. The latter is exactly the space of even functions.

So we study the trace formulas for even functions g. Write $\lambda_j = \frac{1}{4} + r_j^2$. The spectral side is the sum of its Fourier transform on r_j. The geometric side is more complicated, involving a main term (associated with the trivial conjugacy class $\{e\}$) containing the area and some higher order terms coming from closed geodesics (associated with nontrivial-conjugacy classes). If we choose carefully the function g, the higher order terms vanish. We choose carefully a function g with certain growth conditions and consider scalings of g. The trace formulas for g_t produce Weyl's law. Details are in [Whi1] or [LV]. □

If Γ is not cocompact, a brief discussion is in [Lap].

1.4.2. *Jacquet-Langlands correspondence.* Suppose G is an inner form of G', their L-groups are canonically isomorphic. Langlands functoriality predicts a correspondence between automorphic representations of these groups. Jacquet-Langlands establishes (part of) such correspondence. The proof relies on a comparison of the trace formulas for different groups.

First, let us consider some representations of two simple groups over $\mathbb{R}$.

- $SO(3)$. We already know the representations of $SU(2)$. We have the standard representation V, and all irreducible representations of $SU(2)$ are of the form $\mathrm{Sym}^k V$ for some non-negative integer k. we also know that $SU(2)$ is isomorphic to the spin group $Spin(3)$ ([BD]). It is therefore a double covering of $SO(3)$

with kernel $\{\pm I\}$. Therefore, the representations of $SO(3)$ are exactly the representations of $SU(2)$ such that $-I$ acts as identity. The representations $\mathrm{Sym}^n V$ with even k descend to representations of $SO(3)$, and these are all irreducible representations of $SO(3)$. In summary, the representations of $SO(3)$ are parameterized by the dimensions n which must be odd. We call them π_n.

- $PGL(2,\mathbb{R})$. We consider discrete series representations [6] of $PGL(2,\mathbb{R})$. We have discrete series representations $D_k^{\pm}(k \geq 1)$ of $SL(2,\mathbb{R})$. We consider the induced representations

$$D_k = \mathrm{Ind}_{SL(2,\mathbb{R})}^{SL^{\pm}(2,\mathbb{R})} D_k^+ = \mathrm{Ind}_{SL(2,\mathbb{R})}^{SL^{\pm}(2,\mathbb{R})} D_k^-.$$

  They are irreducible unitary representations of $SL^{\pm}(2,\mathbb{R})$, and the restriction to $SL(2,\mathbb{R})$ is $D_k^+ \oplus D_k^-$. Again, $SL^{\pm}(2,\mathbb{R})$ is a double covering of $PGL(2,\mathbb{R})$ with kernel $\{\pm I\}$. So the discrete series D_n with odd n define discrete series representations of $PGL(2,\mathbb{R})$. We call them σ_n. In particular, σ_1 is the Steinberg representation.

Therefore, we have a naive correspondence between representations of $SO(3)$ and discrete series of $PGL(2,\mathbb{R})$. But we could say more, their characters are related by the equation [7]

$$\chi_{\pi_n}\left(\begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{pmatrix}\right) = -\chi_{\sigma_n}\left(\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\right), \theta \in [0, 2\pi].$$

This is the Jacquet-Langlands correspondence over $\mathbb{R}$.

What happens if we are considering representation theory over a nonarchimedean local field F? Recall that over such an F there exists a unique quaternion algebra D with center F. The multiplicative group D^* is an inner form of $G = GL(2,F)$. Moreover, the *conjugacy classes* of D^* correspond bijectively to the *elliptic conjugacy classes* of G. The correspondence is defined by forcing the reduced trace and reduced norm of $x \in D^*$ equal to the trace and determinant of $g \in GL(2,F)$[8]. The local Jacquet-Langlands correspondence asserts a bijection of between the equivalence classes of the irreducible representations of D^* and those of the discrete series of G [9]. If π maps to σ under this correspondence then $\chi_\pi(x) = -\chi_\sigma(g)$ whenever the conjugacy classes of x and g correspond.

Now we can consider the glocal case. Recall that a quaternion algebra over $\mathbb{R}$ are in one-to-one correspondence with finite subsets of places of $\mathbb{Q}$ with the even cardinality. This correspondence is achieved by assigning to D the set S of places where it is ramified. Let D (and therefore S) be given. Set $G' = D^*$, and let $G'(\mathbb{A})^1$ be the elements of

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[6] Nontrivial finite-dimensional representations of $PGL(2,\mathbb{R})$ are not unitary.

[7] Note that $SO(3)$ is compact, so any element is conjugate to an element in the maximal torus. But this does not hold true for $PGL(2,\mathbb{R})$.

[8] Note that the invariants are defined over conjugacy classes. For $GL(2,F)$, the trace and determinant determine a semisimple conjugacy class, thus determining a unique elliptic conjugacy class.

[9] There are two types of discrete series representations. One consists of the supercuspidal representations, their matrix coefficient are compactly supported. Another consists of the special representations, obtained from certain reducible principal series representations.

$G(\mathbb{A})$ with reduced norm 1. $G'\backslash G'(\mathbb{A})^1$ is compact and therefore $L^2(G'\backslash G'(\mathbb{A})^1)$ decomposes discretely into a direct sum of irreducible representations. The one-dimensional constituents are the characters $\chi \circ Ndr$ where χ is a Dirichlet character of $\mathbb{Q}^*\backslash\mathbb{A}^*$. They naturally correspond to the one-dimensional representations $\chi\circ\det$ of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})^1)$.

Theorem 1.9 (Global Jacquet-Langlands)**.**

- *$L^2(G'\backslash G'(\mathbb{A})^1)$ is multiplicity free.*
- *Suppose that $\sigma = \otimes_v \sigma_v$ is an irreducible constituent which is not one-dimensional. Define $\pi = \otimes_v \pi_v$ if $v \notin S$ and π_v corresponds to σ_v under the local Jacquet-Langlands correspondence if $v \in S$. Then π is a cuspidal representation of $G(\mathbb{A})$.*
- *Conversely, any cuspidal representation $\pi = \pi_v$ of $G(\mathbb{A})$ such that π_v is square-untegrable for all $v \in S$ is obtained from an automorphic representation of G' by the above procedure.*

The proof relies on a comparison between the trace formulas for the two groups. To make such a comparison, we first have to define a transfer of functions over the two spaces. We may assume $f' = f'_v$ is decomposable, we need to define a function f. There is a naive identification outside S. If $v \in S$, we choose f_v as a 'zero extension' of f'_v: its orbital integral over elliptic conjugacy classes is the same as that of f'_v under the correspondence of conjugacy classes, and its orbital integral over other classes are zero.[10] Then we need to compare the traces of f' and f. If we consider *all* compactly supported functions, we have to consider continuous spectrum. However, we only need to consider the image of f'_v under the transfer. The vanishing condition guarantees that all the supplementary terms vanish[11]. Therefore, we get an equation

$$\sum_{\pi' \in \prod(G')} m(\pi, R')\mathrm{tr}(\pi(f')) = \sum_{\pi^* \in \prod(G)} m(\pi, R_{disc})\mathrm{tr}(\pi(f)).$$

This is an example of the *simple trace formula*. Since the trace formulas are the same for all f', the only possibility is that all multiplicities of the corresponding representations are the same.

Remark 1.6. There are many generalizations of Jacquet-Langlands correspondence. The most straightforward one is the generalization to $GL(n)$.

There is a geometric approach to the local Jacquet-Langlands correspondence by Yoichi Mieda using the ℓ-adic étale cohomology of the Drinfeld twoer ([Mie]).

Remark 1.7. The simplest application should be the computation of dimensions of modular forms. ([JL2])

In the case $v_\infty \notin V$, it implies a correspondence between spectra of Laplacians on certain compact Riemann surfaces, and discrete spectra of Laplacians on non-compact surfaces.

Jacquet-Langlands correspondence can also be used to define non-isometric spaces with the same spectrum decomposition, giving a negative answer to the question: Can one hear the shape of a drum? ([Kac]).

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[10] Of course, we need to show the existence of such functions. That is, the function defined over conjugacy classes lifts too a function over the whole space.

[11] The bad behavior of trace formula comes from parabolic subgroups.

We may also construct a counterexample as follows. Let Λ be an even unimodular form in $\mathbb{R}^m$, then the theta series associated with Λ is a modular form for $SL(2,\mathbb{Z})$ of weight $m/2$. The eigenvalues of the Laplacian of $\mathbb{R}^m/\Lambda$ are nonnegative integers n with multiplicities the Fourier coefficients of the associated theta series. Now we know $m|8$. And we may find two non-isomorphic even unimodular lattices Λ_1 and Λ_2 in $\mathbb{R}^{16}$. But $\dim M_8(SL(2,\mathbb{Z}))=1$. So the Riemannian manifolds $\mathbb{R}^{16}/\Lambda_1$ and $\mathbb{R}^{16}/\Lambda_2$ are non-isomorphic but have the same spectral decomposition.([BGHZ])

2. The invariant trace formula

The original trace formula is the equality of two expressions of the distribution $J(f)$. As we have seen, J is not invariant under conjugation, so is easy to use in practice. We always assume that we want to study spectral decompositions and therefore only conjugation-invariant objects are interested. So we need a refined version of the trace formula. We need to modify the distribution J to get a conjugation-invariant distribution $I(f)$. And the *stable trace formula* is an equality of two expressions of the new distribution $I(f)$. Another advantage of the stable trace formula is that we have explicit formulas to compute the local contributions over arbitrary conjugacy classes and characters, they are weighted sums of weighted orbital integrals or weighted characters.

2.1. The fine geometric and spectral expansions.

Theorem 2.1. *For any $f\in\mathcal{H}(G)$, $J(f)$ has a geometric expansion*

$$J(f)=\lim_S\sum_M\frac{|W_0^M|}{|W_0^G|}\sum_{\gamma\in\Gamma(M)}a^M(\gamma)J_M(\gamma,f),$$

and a spectral expression

$$J(f)=\lim_T\sum_M\frac{|W_0^M|}{|W_0^G|}\int_{\Pi(M)}a^M(\pi)J_M(\pi,f)d\pi.$$

2.2. **The invariant trace formula.** In order to use the trace formula to study representations, we need to modify the distribution J to make it invariant. The invariant distribution $I(f)$ is defined inductively as

$$I(f)=J(f)-\sum_{L\in\mathcal{L},L\neq G}|W_0^L||W_0^G|^{-1}\hat{I}^L(\phi_L(f)).$$

The distribution I is a linear combination of distributions indexed by Levi subgroups of G, assuming the existence of invariant distributions on all Levi subgroups. In the above definition, ϕ_L is a continuous linear transformation from $\mathcal{H}_{ac}(G)$ to $\mathcal{H}_{ac}(L)$, $\hat{I}$ is the invariant linear form induced from the distribution I^L on $\mathcal{H}_{ac}(L)$.

The invariant trace formula is the equation of two expansions of the distribution $I(f)$. We need to construct an invariant version of linear forms $J_M(\gamma,f)$ and $J_M(\pi,f)$.

Theorem 2.2. *For any $f\in\mathcal{H}(G)$, $I(f)$ has a geometric expression*

$$I(f)=\lim_S\sum_M\frac{|W_0^M|}{|W_0^G|}\sum_{\gamma\in\Gamma(M)}a^M(\gamma)I_M(\gamma,f),$$

and a spectral expression

$$I(f) = \lim_T \sum_M \frac{|W_0^M|}{|W_0^G|} \int_{\Pi(M)} a^M(\pi) I_M(\pi, f) d\pi.$$

2.3. **Applications.**

2.3.1. *Inner forms of $GL(n)$.*

2.3.2. *Functoriality and base changes for $GL(n)$.*

3. The stable trace formula

Given an algebraic group G over F. Two elements are in the *stable conjugacy class* if they are conjugate in $G(\overline{F})$. This is a weaker condition than conjugacy, and any stable conjugacy class is a finite union of conjugacy classes.

Example 3.1. *Let $G = SL(2)$ and $F = \mathbb{R}$, the relation*

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

represents conjugacy over $G(\mathbb{C})$ of non-conjugate elements in $G(\mathbb{R})$.

3.0.1. *Why stabilization?* One of the most important application of trace formulas is to study the Langlands functoriality: we transfer test functions, conjugacy classes, and representations from one group to another. If the orbital integrals coincide under this transformation, then the transformation of representations is the desired functoriality correspondence. Known examples include the Jacquet-Langlands correspondence. Let me explain some difficulities in the transformation and explain why we need to stabilize the trace formula.

- *Geometric side: transfer conjugacy classes.* Conjugacy class for GL_N can be expressed in terms of their characteristic polynomials. We can define the transfer of a conjugacy by determining its characteristic polynomial. However, characteristic polynomials only distinguish stable conjugacy.
- *Spectral side: L-packets.* Lnaglands correspondence associates L-packets to L-parameters. The functoriality is only a correspondence of L-packets. Therefore, we need a formula that studies the trace of representations in L-packets. The characters over L-packets are expected to be stable characters.

  Here is an example. Recall that a semisimple Lie group $G(\mathbb{R})$ has a discrete series if and only if it has a compact Cartan subgroup. More generally, a reductive Lie group $G(\mathbb{R})$ has a discrete series if and only if G-has an elliptic torus T_G over $\mathbb{R}$. any strongly regular elements elliptic conjugacy class for $G(\mathbb{R})$ intersects $T_{G,reg}$. Two elements in $T_{G,reg}$ are $G(\mathbb{R})$-conjugate if and only if they lie in the same $W_{\mathbb{R}}$-orbit.

  Let μ be a character of $Z(\mathfrak{g})$, there are exactly $|W_{\mathbb{C}}/W_{\mathbb{R}}|$-discrete series representations with infinitesimal character μ. Their characters can be written as a distribution on $T_{\mathbb{R}}$ (in fact a function on $T_{G,res}$ by Harish-Chardra's theorem). The explicit formula is given in [Sch1], they are sums over $W_{\mathbb{R}}$, so is (only) $G(\mathbb{R})$-invariant. However, the sums of characters in an L-packets of discrete series is a sum over $W_{\mathbb{C}}$-orbits, so is a stable character.

3.1. **Local stabilization.** The field will be local in this subsection unless otherwise stated. We need to study local functoriality.

3.1.1. *Some formal settings.* A special case is the the theory of *endoscopy*. Let G' be an endoscopic group for G. We need to find a characterization of the image of the functorial correspondence $\pi' \to \pi, \pi' \in \Pi(G')$. *The functorial correspondence of representations is dual to a transfer of functions.* So we required to study the *transfer* of functions from $G(F)$ to $G(F')$.

The transfer of functions is based on harmonic analysis. Its domain is the space of test functions on $G(F)$. We take the Hecke algebra $\mathcal{H}(G)$, a convolution algebra that equals $C_c^\infty(G)$ if F is archimedean, but that is the proper subalgebra of functions $f \in C_c^\infty(G)$ that satisfy a supplementary finiteness condition under left and right translation of f by elements in a fixed maximal compact subgroup, if F is archimedean.(need to modify, define it earlier.)

An element $\gamma \in G(F)$ is called *strongly regular* if its centralizer G_γ is a maximal torus in G. For any such γ, we have the associated *invariant orbital integral*

$$f_G(\gamma) = |D(\gamma)|^{1/2} \int_{G_\gamma(F)\backslash G(F)} f(x^{-1}\gamma x)dx$$

for any test function $f \in \mathcal{H}(G)$, where dx is a fixed right invariant measure. We normalize $f_G(\gamma)$ by the Weyl discriminant

$$D(\gamma) = \det((1 - \mathrm{Ad}(\gamma))_{\mathfrak{g}/\mathfrak{g}_\gamma}).$$

We write $\mathcal{I}(G) = \{f_G : f \in \mathcal{H}(G)\}$ for the image of $\mathcal{H}(G)$ under this transform.

The functions in $\mathcal{I}(G)$ also have a spectral interpretation. Any representation $\pi \in \Pi(G)$ has a chaaracter, which can be identified with the linear form

$$\mathrm{tr}(\pi(f)) = \mathrm{tr}\left(\int_{G(F)} f(x)\pi(x)dx\right), f \in \mathcal{H}(G),$$

on $\mathcal{H}(G)$. We set $f_G(\pi\mathrm{tr}(\pi(f))$. It can be shown that either of the two functions $\{f_G(\gamma)\}$ and $\{f_G(\pi)\}$ attached to f determines the other. We can therefore regard any element f_G in $\mathcal{I}(G)$ as a function of either γ or π. It is *invariant* in the sense that it depends only on the conjugacy class of γ or the equivalence class of π. It also remains invariant if its preimage $f \in \mathcal{H}(G)$ is replaced by any conjugate $f^y(c) = f(ycy^{-1})$.

In summary, orbital integral and trace, the two sides of the trace formula, are two transformations from $\mathcal{H}(G)$ to invariant functions. The fact that the spaces are the same indicates that the conjugacy class is dual to representations.

Two strongly regular elements in $G(F)$ are said to be *stably conjugate* if they are conjugate as elements in the group $G(\overline{F})$. For the local field F, there are only finitely many $G(F)$-conjugacy classes γ in any (strongly regular) stable conjugacy class δ. The corresponding sum

$$f^G(\delta) = \sum_\gamma f_G(\gamma), f \in \mathcal{H}(G)$$

of orbital integrals is called the *stable orbital integral* of the given function f at δ. We write $\mathcal{S} = f^G : f \in \mathcal{H}(G)$ for the space of functions of δ obtained in this way.

As we shall see, L-packets arise when we try to find a spectral interpretation for the functions f^G in $\mathcal{S}(G)$ analogous to the values $f_G(\pi)$ of functions f_G in $\mathcal{I}(G)$. In general, a distribution S on $G(F)$ is said to be *stable* if its value at any f depends on g^G. If this is so, the distribution S descends to a linear form $\hat{S}$ on $\mathcal{S}(G)$: $\hat{S}(f^G) = S(f), f \in \mathcal{H}(G)$. The spectral question is then to attach stable distributions to representations π.

3.1.2. *Transfer and the fundamental lemma.* Suppose that G' is an endoscopic datum for G. Langlands and Shelstad define a strongly regular element $\delta' \in G'(F)$ to be *strongly G-regular* if its image in $G(F)$ is strongly regular for G. The space of strongly G-regular elements remains open and dense in $G'(F)$. They introduced an explicit function $\Delta(\delta', \gamma)$ of strongly stable conjugacy classe which they called a *transfer factor* for G and G'. By construction, this function vanishes unless γ belongs to the stable conjugacy class of the image of δ' in $G'(F)$. It therefore has finite support in either of the variables when the complementary variable is fixed. The role of $\Delta(\delta', \gamma)$ is as the kernel function for the transfer mapping that sends a function $f \in \mathcal{H}(G)$ to the function

$$f'(\delta') = f_{\Delta}^{\delta'} = \sum_{\gamma} \Delta(\delta', \gamma) f_G(\gamma)$$

of δ'. Langlands and Shelstad conjectured that the function $f'(\delta')$ belongs to the space $\mathcal{S}(G')$.

The Langlands-Shelstad transfer conjecture is true for archimedean F. For nonarchimedean F, it is closed tie to the *fundamental lemma*. Let G and G' be two unramified group. Then $G(F)$ has a hyperspecial maximal compact subgroup K_F, which is determined uniquely up to the appropriate analogue of stable conjugacy. The fundamental lemma asserts that if f is the characteristic function of K_F, then f' equals the image in $\mathcal{S}'(G')$ of the characteristic function of any hyperspepcial maximal compact subgroup K'_F of $G'(F)$. The fundamental lemma is thus a more precise version of the transfer conjecture in a special case. This is now a theorem by Ngô.

3.1.3. *Local spectral transfer.* The Langlands-Shelstad transfer is a transfer based on conjugacy classes. We also define a spectral transfer.

3.2. **The global stabilization.**

3.2.1. *The problem of stabilization.* Remember that we want to find stably invariant distributions. We may, of course, sum over the conjugacy classes in the same stable conjugacy class to make an invariant distribution stable since there are only finitely many conjugacy classes in a stable conjugacy class. This is what we have done for local fields.

Could we do the same thing for a global field to get something stable? The analysis of global fields is the analysis of functions over the locally compact group $G(\mathbb{A}_F)$. Let

$$I_{reg,ell}(f) = \sum_{\gamma \in \Gamma_{reg,ell}} a^G(\gamma) f_G(\gamma)$$

where $a^G(\gamma) = \mathrm{vol}(G_\gamma(\mathbb{Q}) \backslash G_\gamma(\mathbb{A})^1)$, f_G is the orbital integral distribution. We might hope that the distribution $I_{reg,ell}$ is stable since we have sum over stable conjugacy classes. However, this not the correct. the stable distribution should be the tensor product of local stable distributions $\{f_v^G(\delta_v)\}$. However, there are not enough rational conjugacy classes to cover the tensor products of local rational conjugacy classes.

Now F is global. Even if we sum over stable conjugacy classes, the orbital integral is still not invariant under $G(\mathbb{A})$. So we need some stabilization.

Let $\gamma \in G(F)$ be a semi-simple element, and let G_γ be its centralizer. If $\gamma' = g^{-1}\gamma\gamma(g \in G(\overline{F}))$ is in a stable conjugacy class. Let $\sigma \in \Gamma_F = \mathrm{Gal}(\overline{F}/F)$, the map $\sigma \mapsto g\sigma^{-1}$ defines a one-cocycle in $H^1(\mathrm{Gal}(\overline{F}/F), G_\gamma(\overline{F}))$. The set of conjugacy classes is parameterized by the abelian group $H^1(\mathrm{Gal}(\overline{F}/F), G_\gamma(\overline{F}))$.

However, each $G(\mathbb{A})$-conjugacy class in the $G(\mathbb{A})$-stable conjugacy class does not necessarily have a representative in $G(F)$. The failure can be measured using class field theory

$$\mathrm{coker}^1(F,T) = \mathrm{coker}(H^1(F,T) \to \bigoplus_v H^1(F_v,T)).$$

So we need to add error terms coming from this.

$$I_{reg,ell}(f) = \sum_{\delta \in \Delta_{reg,ell}(G)} a^G(\delta)\iota(T) \sum_{\kappa \in \hat{T}^\Gamma} f_G^\kappa(\delta),$$

3.2.2. *Endoscopic datum.* Surprisingly, the error terms have a group theoretical interpretation. This is the *endoscopic datum.* Let G be an algebraic group, an endoscopic datum for G is defined to be a triplet $(G', \mathcal{G}', s', \xi')$, where G' is a quasi-split group over F, $\mathcal{G}'$ is a split extension of W_F by a dual group $\hat{G}'$ of G', s' is a semisimple element in $\hat{G}$, and ξ' is an L-embedding of $\mathcal{G}'$ into ${}^LG$. It is required that $\xi'(\hat{G}')$ be equal to the connected centralizer of s' in $\hat{G}$, and that

$$\xi'(u')s' = s'\xi'(u')a(u')$$

where a is a 1-cocycle from W_F to $Z(\hat{G})$ that is locally trivial. We say that $(G'\mathcal{G}', s'\xi')$ is *elliptic* if $Z(\xi')^0 = 1$. The isomorphisms between endoscopic data are defined. This means that the image of ξ' in ${}^LG$ is not contained in ${}^LM$ for any proper Levi subgroup of G over F. We weite $\mathcal{E}_{\mathrm{ell}}(G)$ for the set of isomorphism classes of elliptic endoscopic data for G.

The construction:

Example 3.2. *Suppose that $G = GL(2)$ and that s' is the image of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in $\hat{G} = PGL(2,\mathbb{C})$. Then $\hat{G}_{s',+}$ consists of the group of diagonal matrices, together with a second component generated by the element $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Since the center of $\hat{G}_{s',+}$ equals $\{1, s'\}$, we obtain elliptic endoscopic data for G by choosing nontrivial homomorphisms from Gal_F to the group $\pi_0(\hat{G}_{s',+}) \cong \mathbb{Z}/2\mathbb{Z}$. The classes in $\mathcal{E}_{\mathrm{ell}}$ other than G itself, are thus parameterized by quadratic extensions E of F.*

So we finally get Langlands' stabilization

$$I_{reg,ell}(f) = \sum_{G' \in \mathcal{E}_{ell}(G)} \iota(G,G')\hat{S}'_{G_{reg,ell}}(f')$$

3.3. **The stable trace formula.** The stable invariant distribution should be a linear combination of invariant distributions.

Theorem 3.3 (Endoscopic trace formula)**.** *th*

$$I^G_{disc}(f) = S^G(f) + \sum S^H(f^H).$$

4. Arthur's classification

One application of trace formula is Arthur classification of automorphic forms of quasi-split special orthogonal and symplectic group. I will just give a sketch. The references are [Art1] and [Art3]. They are both summaries of [Art5]. We assume the results for $GL(n)$ are already known. Orthogonal groups and symplectic groups are considered as endoscopic groups of $GL(N)$. In this section, G is a quasisplit orthogonal or symplectic group.

Remark 4.1. The endoscopic classification of unitary groups is already done in [Mok] for quasi-split unitary groups, and in [KWSW] for inner forms of unitary groups.

4.1. **Statement of results.**

4.1.1. *Local results.* Assume F is a local field. We have the local Langlands group L_F. The group G somes with the family $\Phi(G)$ of (conjugace classes of) Langlands parameters, and the family $\Pi(G)$ of (equivalence classes) irreducible admissible representations of $G(F)$. Here we actually take a quotient of these objects. $\tilde{\Phi}(G)$ is the same as $\Phi(G)$ for types B_n and C_n, but is the set of $O(2n,\mathbb{C})/SO(n,\mathbb{C})$-orbits in $\Phi(G)$ under the action of $O(2n,\mathbb{C})$ by conjugation on ${}^LG$. $\tilde{\Pi}(G)$ is the sames as $\Pi(G)$ for types B_n and C_n, but is the set of $O(2n,F)/SO(2n,F)$ orbits in $\Pi(G)$ under the action of $O(2n,F)$ by conjugation on $G(F)$.

We have a chain

$$\tilde{\Pi}(G)_{\text{temp}}G) \subset \tilde{\Pi}_{\text{unit}}(G) \subset \tilde{\Pi}(G),$$

we also define a chain

$$\tilde{\Phi}_{\text{bdd}}(G) \subset \tilde{\Psi}(G) \subset \tilde{\Phi}(G)$$

defined by certain bounded conditions.

For any $\psi \in \tilde{\Psi}(G)$, we have the centralizer S_ψ in $\hat{G}$ of the image of ψ. Let $\overline{S}_\psi = S_\psi/Z(\hat{G})^{\text{Gal}_F}$. The group $\mathcal{S}_\psi = \pi_0(\bar{(}S_\psi))$ is a finite abelian 2-group.

Roughly speaking, the classification theorem asserts that we may associate a packet $\tilde{\pi}_\psi$ over $\tilde{\Pi}_{\text{unit}}(G)$ to a parameter $\psi \in \tilde{\Psi}(G)$. The packet $\tilde{\pi}_\psi$ is parameterized by characters of $\mathcal{S}_\psi$. If ψ is bounded, the packets can be regarded as a packet of tempered representations. These tempered representations exhausts the set $\tilde{\Pi}_{\text{temp}}(G)$. For the accurate statement, see [Art1].

4.1.2. *Global results.* Now F is a global field. Assuming the existence of the global Langlands group L_F, we define the global parameter sets $\tilde{\Phi}_{\text{bdd}}(G)$, $\tilde{\Psi}(G)$ and $\tilde{\Phi}(G)$ as in the local case. To a parameter $\psi \in \tilde{\Psi}(G)$, we also have the centralizer S_ψ and the associated 2-group $\mathcal{S}_\psi$. For any v, we have the localization mapping $x \to x_v$ from $\mathcal{S}_\psi$ to $\mathcal{S}_{\psi_v}$. The global packet is defined as

$$\tilde{\Pi}_\psi = \{\pi = \bigotimes \pi_v : \pi_v \in \tilde{\Pi}_{\psi_v}, \langle \cdot, \pi_v \rangle = 1 \text{ for almost all} v\}.$$

The main global theorem is a decomposition of the automorphic discrete spectrum

$$L^2_{\text{disc}}(G(F)\backslash G(\mathbb{A})) \subset L^2(G(F)\backslash G(\mathbb{A})).$$

The space $L^2_{\text{disc}}(G(F)\backslash G(\mathbb{A}))$ is a Hecke algebra module. We need to consider a slightly smaller Hecke algebra $\tilde{\mathcal{H}}(G)$ as we only consider orbits of representations for type D_n. The accurate statement and the explanations of symbols can be find in [Art1].

4.2. **The idea of classification.** we sketch the procedure of Arthur classification. The basic idea is endoscopy. We already know that the L-group of G can be canonically embedded as a subgroup of the L-group of $GL(N)$. We already know the Langlands correspondence of $GL(N)$. If the L-parameter factors through a subgroup associated with G, it should corresponds a L-packet of representations of G by Langlands functoriality. Such parameters are called the *Arthur parameter.*

4.2.1. *Self-dual representations and endoscopic subgroups.* Let $\tilde{G} = GL(N)$ over F, equipped with the outer automorphism $\alpha : x \mapsto x^\vee :=^t x^{-1}$. The corresponding complex dual $\hat{\tilde{G}} = GL(N, \mathbb{C})$ comes with the dual outer automorphism $\hat{\alpha} : g \to g^\vee$.

Assuming the existence of the global langlands group L_F, we consider the self-dual $\hat{\alpha}$-stable and $\hat{\alpha}$-discrete continuous homomorphisms

$$\psi : L_F \times SL(2, \mathbb{C}) \to \hat{\tilde{G}} = GL(n, \mathbb{C}).$$

By definition,

$$\psi = \psi_1 \oplus \cdots \oplus \psi_r,$$

for distinct irreducible representations

$$\psi_i : L_F \times SL(2, \mathbb{C}) \to GL(N_i, \mathbb{C}),$$

such that ψ_i is equivalent to $\psi_j^\vee$. ψ is called *elliptic* if $\psi_i^\vee = \psi_i$ for each i. From now on, we always assume that ψ is elliptic. Then ψ_i is either *symplectic* or *orthogogal*, which means that its image is contained in the subgroup $Sp(N_i, \mathbb{C})$ or $O(N_i, \mathbb{C})$, up to conjugacy.

Writing $\psi_i = \mu_i \otimes \nu_i$ for irreducible representations $\mu_i : L_F \to GL(m_i, \mathbb{C})$ and $\nu_i : SL(2, \mathbb{C}) \to GL(n_i, \mathbb{C})$ such that $N_i = m_i n_i$. Then for any i, μ_i is equivalent to $\mu_i \otimes \chi$. ν_i is symplectic or orthogonal according to whether it is even or odd dimensional. ψ_i is symplectic if and only if one of μ_i and ν_i is symplectic, another is orthogonal. ψ_i is orthogonal if both are orthogonal or symplectic.

Collecting all orthogonal and symplectic factors together, we see that the image of ψ is in a subgroup $O(N_O, \mathbb{C}) \times Sp(N_S, \mathbb{C}) \subset GL(N, \mathbb{C})$. Therefore, the elliptic self-dual representation factors through the embedding subgroup

$$^L G_{E/F} =^L (G_O \times G_S)_{E/F}$$

of $SL(N,\mathbb{C})$ attached to the quasisplit group

$$G = G_O \times G_S$$

over F. The group G is called a *twisted endoscopic group* for $GL(N)$. From Langlands functoriality, this parameter should determine a packet of automorphic representations of G. In particular, if G is just a orthogonal or symplectic group, we should get automorphic representations of G.

4.2.2. *Arthur parameter.* The existence of L_F is only hypothetical now. Even if it does exist, an explicit description of representations ψ is useful. The main hypothetical property of L_F is that its irreducible unitary representations of dimension m_i should be in canonical bijection with the unitary cuspidal representations of $GL(m)$. So we could have a representation theoretical interpretation of ψ.

A unitary cuspidal automorphic representation μ of $GL(m)$ is called *χ-self-dual* if the representation $x \to \mu(x^\vee)$ is equivalent to μ. These are exactly automorphic representations associated with self-dual L-parameters. We also need to define what is an orthogonal or symplectic automorphic representation. In general, this must be done in terms of whether a certain automorphic L-function for μ has a pole at $s = 1$. See [Art1], Theorem 3.

4.3. **An example:** $GSp(4)$**.** A slightly different example is $GSp(4)$ ([Art2]). The abstract classification theorem becomes explicit in this simple case. The classification is parallel to the classification theorem above. I will just say something that is particular for $GSp(4)$.

Since we are considering $GSp(4)$, we need a more complex torus factor. So let $\tilde{G} = GL(N) \times GL(1)$ over F, equipped with the outer automorphism

$$\alpha : (x, y) \mapsto (x^\vee, \det(x)y)$$

The corresponding complex dual $\hat{\tilde{G}} = GL(N,\mathbb{C}) \times \mathbb{C}^*$ comes with the dual outer automorphism

$$\hat{\alpha} : (g, z) \to (g^\vee z, z).$$

The self-dual $\hat{\alpha}$-stable and $\hat{\alpha}$-discrete continuous homomorphisms are then

$$\tilde{\psi} = \psi \oplus \chi : L_F \times SL(2,\mathbb{C}) \to \hat{\tilde{G}} = GL(n,\mathbb{C}) \times \mathbb{C}^*.$$

By definition,

$$\psi = \psi_1 \oplus \cdots \oplus \psi_r,$$

for distinct irreducible representations

$$\psi_i : L_F \times SL(2,\mathbb{C}) \to GL(N_i,\mathbb{C}).$$

We assume $\tilde{\psi}$ is elliptic so that ψ_i is equivalent to $\psi_i^\vee \otimes \chi$. Then ψ_i is either *symplectic* or *orthogogal*, which means that its image is contained in the subgroup $GSp(N_i,\mathbb{C})$ or $GO(N_i,\mathbb{C})$, up to conjugacy. Writing $\psi_i = \mu_i \otimes \nu_i$ for irreducible representations $\mu_i : L_F \to GL(m_i,\mathbb{C})$ and $\nu_i : SL(2,\mathbb{C}) \to GL(n_i,\mathbb{C})$ such that $N_i = m_i n_i$. Then for any i, μ_i is equivalent to $\mu_i \otimes \chi$. ν_i is symplectic or orthogonal according to whether it is even or odd dimensional. ψ_i is symplectic if and only if one of μ_i and ν_i is symplectic, another is orthogonal. ψ_i is orthogonal if both are orthogonal or symplectic.

$$\hat{G} = \{(g_+, g_-, z) \in \hat{G}_+ \times \hat{G}_- \times \mathbb{C}^* : \lambda(g_+) = \Lambda(g_-) = z\}$$

determines

The image is again contained in the subgroup

$$\hat{G} = \{(g_+, g_-, z) \in GO(N_+, \mathbb{C}) \times GSp(N_-, \mathbb{C}) \times \mathbb{C}^* : \lambda(g_+) = \Lambda(g_-) = z\}$$

of a direct product of $GO(N_+, \mathbb{C})$ and $GSp(N_-, \mathbb{C})$. These generalized orthogonal or symplectic groups are dual groups of some quais-split groups G_+ and G_-. Then the subgroup $\hat{G}$ is the L-group of a quotient group G of $G_+ \times G_-$, called an *endoscopic group*.

If we assume that N is even, then G is isomorphic to the general spin group $GSpin(N+1) = (Spin(N+1) \times \mathbb{C}^*)/\{\pm 1\}$ over F. If $N = 4$, there is an exceptional isomorphism between $GSpin(5)$ and $GSp(4)$. This is why we could study the automorphic representations of $GSp(4)$.

The existence of L_F is only hypothetical now. Even if it does exist, an explicit description of representations ψ is useful. The main hypothetical property of L_F is that its irreducible unitary representations of dimension m_i should be in canonical bijection with the unitary cuspidal representations of $GL(m)$. So we could have a representation theoretical interpretation of $\tilde{\psi}$.

A unitary cuspidal automorphic representation μ of $GL(m)$ is called χ-*self-dual* if the representation

$$x \to \mu(x^\vee)\chi(\det x)$$

is equivalent to μ. We also need to define what is an orthogonal or symplectic automorphic representation. In general, this must be done in terms of whether a certain automorphic L-function for μ has a pole at $s = 1$. If $m = 2$ or 4, we have explicit descriptions ([Art4]).

Let $G = GSpin(N+1)$. Define the *Arthur packet* $\Psi_2(G, \chi)$ to be the set of formal (unordered) sums

$$\psi = \psi_1 \boxplus \cdots \boxplus \psi_r$$

of distinct, formal, χ-self dual tensor products

$$\psi_i = \mu_i \boxtimes \nu_i, \quad 1 \leq i \leq r,$$

of symplectic type. More precisely, ν_i is an irreducible representation of $SL(2, \mathbb{C})$ of dimension m_i, and μ_i is a χ-self dual, unitary, cuspidal automorphic representation of $GL(m_i)$ that is of symplectic type if n_i is odd and orthogonal type if n_i is even,, for integers m_i and n_i such that

$$N = N_1 + \cdots + N_r = m_1 n_1 + \cdots + m_r n_r.$$

The correspondence should be that these Arthur packets has a natural correspondence with automorphic representations. A character of $\Gamma_F$is equivalent to a Grossencharacter by class field theory. $\Pi_2(G, \chi)$ corresponds to $\Psi_2(G, \chi)$, the unitary cuspidal automorphic representations in $L^2_{disc}(G(\mathbb{F})\backslash G(\mathbb{A}), \chi)$.

The explicit classification of $GSp(4)$ is as follows:

(1), *General type.*

$$\psi = \psi_1 = \mu \boxtimes 1.$$

(2). *Yoshida type.*

$$\psi = \psi_1 \boxplus \psi_2 = (\mu_1 \boxtimes 1) \boxplus (\mu_2 \boxtimes 1).$$

(3). *Soudry type.*

$$\psi = \psi_1 = \mu \boxtimes \nu(2).$$

(4). *Saito-Kurakawa type*

$$\psi = \psi_1 \boxplus \psi_2 = (\lambda_1 \boxtimes \nu(2)) \boxplus (\mu \boxplus 1).$$

(5). *Howe-Piatetski-Shapiro type*

$$\psi = \psi_1 \boxplus \psi_2 = (\lambda_1 \boxtimes \nu(2)) \boxplus (\lambda_2 \boxtimes \nu(2)).$$

(6). *One-dimensional type*

$$\psi = \psi_1 = \lambda \boxtimes \nu(4).$$

References

[Art1] J. Arthur, *The Endoscopic Classification of Representations*, in *Automorphic Representations and L-functions*, Tate Institute of Fundamental Research. 2013, 1-22.

[Art2] J. Arthur, *Automorphic Representations of* $GSp(4)$

[Art3] J. Arthur, *Classifying automorphic representations*

[Art4] J. Arthur, *An introduction to the trace formula*

[Art5] J. Arthur, *The Endoscopic classification of Representations: Orthogonal and Symplectic Groups*, Colloquium Publications, 61,2013, American Mathematical Society.

[BD] Theodor Bröcker and Tammo tom Dieck, *Representations of Compact Lie Groups,* GTM 98.

[BGHZ] Jan Hendrik Bruinier, Gerard van der Geer, Günter Harder and Don Zagier, *The 1-2-3 of Modular Forms.*

[Cha] Pierre-Henri Chaudouard, *A short introduction to the trace formula.*

[Gel] Steven Gelbart, *Lectures on the Arthur-Selberg trace formula*

[Har] Michael Harris, *Introduction to the stable trace formula.*

[JL] H. Jacquet and R. Langlands, *Automorphic Forms on* $GL(2)$, Springer Lecture Notes in Mathemstics. **144** (1970).

[JL2] *The Jacquet Langlands correspondence for* GL_2. Lecture note 20.

[Kac] Mark Kac, *Can one hear the shape of a drum?*

[KWSW] Tasho Kaletha, Alberto Minguez, Sug Woo Shin, and Paul-James White, *Endoscopic classification of representations: inner forms of unitary groups.*

[Kna3] A.W. Knapp, *Theoretical aspects of the trace formula for* $GL(2)$.

[Kna5] A. Knapp, *Representations of* $GL_2(\mathbb{R})$ *and* $GL_2(\mathbb{C})$, Proceedings of Symposium in Pure Mathematics, vol **33**, part I, 87-91.

[Lab] J.P. Labesse, *The Langlands spectral decomposition*

[Lan1] Robert P. Langlands, *The trace formula and its applications: an introduction to the work of James Arthur.*

[Lap] Erez M. Lapid, *Introductory Notes on the Trace Formula.*

[LV] Elon Lindenstrauss and Akshay Venkatesh, *Existence and Weyl's law for spherical cusp forms.*

[Mie] Yoichi Mieda, *Geometric approach to the local Jacquet-Langlands correspondence.*

[Mok] Chung Pang Mok, *Endoscopic classification of representations of quasi-split unitary groups.*

[Mon] Gerard Greixas i Montplet, *The Jacquet-Langlands correspondence and the arithmetic Riemann-Roch theorem for pointed curves.*

[Sch1] W. Schmid, *Discrete Series*, Proceedings of Symposium in Pure mathematics, vol **61**(1997), 83-113

[Sch2] W. Schmid, *Geometric methods in representation theory.*

[Whi1] David Whitehouse, *An introduction to the trace formula.*