Hodge Classes in the Cohomology of Local Systems

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Hodge Conjecture

Conjecture (Hodge conjecture)

Let X be a non-singular complex projective manifold. Then every Hodge class on X is a linear combination with rational coefficients of the cohomology classes of complex subvarieties of X.

The codimension-1 case is the Lefschetz theorem on $(1, 1)$ classes. This (and its dual) is the only known case in general.

Some strategies

- Inspired by Lefschetz's original approach, Griffiths-Green program studies normal functions arising from fibering a Hodge class out over a base. In particular, they study the singularities of (admissible) normal functions.
- Voison pointed out that one could break the Hodge conjecture into a question about the absoluteness of Hodge classes and the Hodge conjecture for varieties defined over $\overline{\mathbb{Q}}$.

Abelian Varieties

The Hodge conjecture holds for

- (Mattuck) a general abelian variety,
- (Tate) an abelian variety A which is isogenous to a product of elliptic curves,
- (Tankeev) a simple abelian variety X whose dimension is a prime number,
- (Schoen) four-dimensional abelian variety of Weil type with $K = \mathbb{Q}(\mu_3)$ or $\mathbb{Q}(\sqrt{-1})$.

Theorem (Deligne)

Let A be an abelian variety over an algebraically closed field k, then every Hodge class is an absolute Hodge class.

Some other results

- Special varieties: hypersurfaces of degree one and two, (Zucker) cubic fourfolds, (Murre) unirational fourfolds, intersections of low degree hypersurfaces; (Shioda) certain Fermat varieties.
- Families of varieties: (D.Arapura) universal curves of stable genus two curves.
- Locally symmetric varieties: (Bergeron, Millson, Moeglin, Zhiyuan Li) arithmetic ball quotients, arithmetic manifolds of orthogonal type, moduli space of quasi-polarized $K3$ surfaces.

We are mainly interested in families of varieties.

Decomposition Theorem

Let X be a complex projective manifold. To any pair (U, L) where $j : U \to X$ is a Zariski open subset of X and L a local system over U, we can canonically define a perverse sheaf *intersection complex* $IC_U(L)$. The decomposition theorem studies the topological properties of proper maps between algebraic varieties.

Theorem (Decomposition theorem)

Let $f : Y \to X$ be a proper map of complex algebraic varieties. There exists an isomorphism in the constructible bounded derived category $\mathcal{D}_X$:

$$Rf_* IC_Y \cong \bigoplus_i {}^{\mathfrak{p}}\mathcal{H}^i(Rf_* IC_Y)[-i].$$

Furthermore, the perverse sheaves ${}^{\mathfrak{p}}\mathcal{H}^i(Rf_* IC_Y)$ are semisimple; i.e.,

$${}^{\mathfrak{p}}\mathcal{H}^i(Rf_* IC_Y) \cong \bigoplus_\alpha IC_{\overline{X_\alpha}}(L_\alpha).$$

Decomposition Theorem

Taking hypercohomology of the decomposition theorem, we get:

Theorem (Decomposition theorem)

Let $f : Y \to X$ be a proper map of varieties. There exists finitely many triples $(X_\alpha, L_\alpha, d_\alpha)$ made of locally closed, smooth and irreducible algebraic subvarieties $X_\alpha \subset X$, semisimple local systems L_α on X_α and integer numbers d_α, such that for every open set $U \subset X$ there is an isomorphism

$$IH^r(f^{-1}U) \cong \bigoplus_\alpha IH^{r-d_\alpha}(U \cap \overline{X}_\alpha, L_\alpha).$$

Remark

The decomposition is not uniquely defined. But in the case when X is quasi-projective, one can make distinguished choices that realize the summands as mixed Hodge substructures of a canonical mixed Hodge structure on $IH^*(Y)$.

Zucker's Conjecture

Let G be a semisimple algebraic group defined over $\mathbb{Q}$ of Hermitian type with the associated Hermitian symmetric domain $D = G(\mathbb{R})/K$. Let Γ be an arithmetic subgroup of G and let $X := \Gamma\backslash D = \Gamma\backslash G(\mathbb{R})/K$ be a locally symmetric variety. Then X is a quasi-projective variety with Baily-Borel compactification $\overline{X}$. Denote by $i : X \to \overline{X}$ the natural inclusion map. Let (V, ρ) be a (rational) representation of G, it defines a local system $\mathbb{V}$ over X. We are interested in the intersection cohomology $IH^*(\overline{X}, \mathbb{V})$.

The Hermitian symmetric domain D is equipped with a canonical Riemannian metric induced from the Killing form of the Lie algebra $\mathfrak{g}$. This metric is Γ-invariant, thus descends to a Riemannian metric over $X = \Gamma\backslash D$. We also choose and fix a metric on the local system $\mathbb{V}$. The L^2-cohomology groups $H^*_{(2)}(X, \mathbb{V})$ are defined to be the cohomology groups of the complex $(C^\bullet, d)$, where C^k is the space of $\mathbb{V}$-valued smooth k-forms over X such that the form itself and its exterior derivative are both square-integrable; the differential map d is simply the restriction of the usual exterior differential.

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Zucker's conjecture

Zucker's conjecture compares the intersection cohomology over $\overline{X}$ and L^2-cohomology over X. It was proved in different ways by Eduard Looijenga and by Leslie Saper and Mark Stern.

Theorem (Zucker's conjecture)

As real vector spaces, the intersection cohomology $IH^*(\overline{X}, \mathbb{V}_{\mathbb{R}})$ is isomorphic to the L^2-cohomology $H^*_{(2)}(X, \mathbb{V}_{\mathbb{R}})$.

Remark

It is natural to ask whether Zucker's conjecture is an isomorphism of Hodge structures. This is an open question.

Relative Lie algebra cohomology

Let G be a reductive Lie group with complexified Lie algebra $\mathfrak{g}$. Fix a maximal compact subgroup K, let U be a $(\mathfrak{g}, K)$-module. The relative Lie algebra cohomology $H^*(\mathfrak{g}, \mathfrak{k}, U)$ is defined to be the cohomology of the complex $(C^\bullet, d)$ where $C^k = \mathrm{Hom}(\wedge^k(\mathfrak{g}/\mathfrak{k}), U)$. The differential map d is defined as in differential geometry as we can regard elements in Lie algebras as invariant tangent vectors.

Let V be a finite dimensional representation, a $(\mathfrak{g}, K)$-module U is called cohomological with respect to V if $H^*(\mathfrak{g}, K.U \otimes V) \neq 0$.

Relative Lie Algebra Cohomology

The L^2-cohomology groups can be interpreted as relative Lie algebra cohomology groups:

$$H^*_{(2)}(X, \mathbb{V}) \cong H^*(\mathfrak{g}, K, L^2(\Gamma \backslash G(\mathbb{R}))^\infty \otimes V).$$

the L^2-cohomology groups remain unchanged if we replace $L^2(\Gamma \backslash G(\mathbb{R}))^\infty$ with $\mathcal{A}^2(G; \Gamma)$, the subspace of L^2-automorphic forms:

$$IH^*(\overline{X}, \mathbb{V}) = H^*(\mathfrak{g}, K, \mathcal{A}^2(G; \Gamma) \otimes V) = \bigoplus_{U_\pi \in \mathcal{A}^2(G;\Gamma)} m_\pi(\Gamma) H^*(\mathfrak{g}, K, U_\pi \otimes V).$$

Hodge representations

If $\mathbb{V}$ comes from geometry, $IH^*(\overline{X}, \mathbb{V})$ is a Hodge structure. We are interested in the possible realizations of local systems as variations of Hodge structures over a locally symmetric variety. Griffiths-Green-Kerr developed the theory of *Hodge representations* to solve this problem.

If D is Hermitian, the relative Lie algebra cohomology $H^*(\mathfrak{g}, K, U_\pi \otimes V)$ has a natural bigrading. To make it compatible with the Hodge structure on the intersection cohomology, we need to do a slightly modification

Vogan-Zuckerman's Classification

D.Vogan and G.Zuckerman classified the cohomological representations in terms of θ-stable parabolic subalgebras. Roughly speaking, cohomological representations should have a minimal K-representation with respect to a Weyl chamber.

Some examples

Example

Discrete series representations are cohomological.

Example

If V is regular, the only cohomological representations are discrete series representations.

Example ($SL_2(\mathbb{R})$)

- Discrete series representations are cohomological;
- Principal series representations are not cohomological.

Some calculations

- For $SU(2,1)$ or $SU(3,1)$, no cohomological representations contributes to Hodge classes.
- For $Sp(4)$, we are intersested in a non-tempered cohomological representation σ_{k+3}. It produces Hodge classes in $H^1(\Gamma\backslash\mathfrak{H}_2, \mathbb{V}^{k,k})$

Question

How to find the multiplicities of σ_{k+3} in $\mathcal{A}^2(\Gamma\backslash\mathfrak{H}_2)$?

Automorphic representations.

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Automorphic representations.

Classification of Automorphic Representations of $GSp(4)$

Arthur classified automorphic representations of $GSp(4)$ in terms of A-packets. There are six types of automorphic representations:

- General type;
- Yoshida type;
- Soudry type;
- Saito-Kurokawa type;
- Howe-Piatetski-Shapiro type;
- one dimensional type.

The representations σ_k are the archimedean components of automorphic representations of the Saito-Kurokawa type.

Multiplicity Formula

We can determine the multiplicity of σ_k for parampdular subgroups of prime level.

Proposition

Let $\Gamma = \Gamma^{\mathrm{para}}(p)$ *be a paramodular subgroup of prime level, then the multiplicity of* σ_k *in* $\mathcal{A}^2(\Gamma \backslash Sp(4, \mathbb{R}))$ *is:*

- $\dim S_{2k-2}(SL(2, \mathbb{Z})) + \dim S_{2k-2}(\Gamma_0(p))^{new,+}$ *if* k *is odd;*
- $\dim S_{2k-2}(\Gamma_0(p))^{new,-}$ *if* k *is even.*

Siegel Modular threefolds

Theorem (D. Arapura)

The Hodge conjecture holds for the universal genus 2 curve.

Theorem

The Hodge conjecture holds for the self-fiber product of the universal genus 2 curve, as well as for the universal abelian surface (and any compactification thereof), over $\Gamma^{\mathrm{para}}(p)\backslash\mathfrak{H}_2$ *when* $p = 1, 2, 3, 5$.

Ball Quotients

Theorem

Let X be a compactification of the universal genus four Picard curve over an arithmetic quotient of $\mathbb{B}^3$. Then the Hodge conjecture holds for X.

Further Questions

Question

For $p = 7$, we have a real Hodge class. Is this a rational class? If so, is there a geometric interpretation of this class?

Question

Could we study paramodular subgroups of all levels at the same time?

Thank You!